Ising-Kac Models near Criticality

by

Massimo Iberti

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

2018

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments

Every piece of the present work contains a story behind it, a story made of people, that can rarely be translated into words and numbers. This thesis would have been impossible without them, because all of them have, directly and indirectly, contributed towards its realization.

I wish to thank my supervisor Martin Hairer for the opportunity he gave me to work on an interesting topic of research. Without his immense patience and his ability this thesis would have never been born.

I am grateful to many people I had the occasion to meet and discuss in Warwick, in particular Sigurd Assing and Nikos Zygouras for numerous meetings and suggestions.

I would like to thank Milton Jara, David Brydges and Jean-Christophe Mourrat for the time dedicated to me in occasion of their visits to Warwick.

I am also very thankful to Hendrik Weber and Giambattista Giacomin for numerous discussions and for agreeing to be the examiners in my defence.

My gratitude goes to all the stochastic analysis group, not only for the friendly environment and helpful comments, but also for their positivity and encouragement throughout the time I spent in Warwick: among them Dirk, Pavlos, Andris, Philipp, Giuseppe, Hao, Ajay, Cyril, Yvain, Frank and Weijun.

A special thank to Elena, for all her support overcoming miles and years, and to my family for being a strong anchor I could rely on in difficult moments.

An finally many thanks to the wonderful friends I had the opportunity to meet during my PhD: I wish all the best to Cassa, Gian, Giovanni, Ilaria, Francesco, Paolo, Aurelio, Alessandro, Jacopo, Ferdinando, Andrea, Ronja, Taisa, Lorenzo, Livia, Alice, Bruno, Mattia, Enrico and Serena.

Declarations

This thesis has been written in partial fulfilments of the requirements for the Degree of Doctor of Philosophy of the the University of Warwick.

I declare that the thesis has been written by myself, it has not been submitted for any other degree or qualification and it is my original work, unless otherwise states and quoted.

Parts of this thesis have been made public [Ibe17] or published [HI18] in a joint work with M. Hairer.

Abstract

The present thesis consists in an investigation around the result shown by H. Weber and J.C. Mourrat in [MW17a], where the authors proved that the fluctuation of an Ising models with Kac interaction under a Glauber-type dynamic on a periodic two-dimensional discrete torus near criticality converge to the solution of the Stochastic Quantization Equation Φ^4_2.
In Chapter 2, starting from a conjecture in [SW16], we show the robustness of the method proving the convergence in law of the fluctuation field for a general class of ferromagnetic spin models with Kac interaction undergoing a Glauber dynamic near critical temperature. We show that the limiting law solves an SPDE that depends heavily on the state space of the spin system and, as a consequence of our method, we construct a spin system whose dynamical fluctuation field converges to Φ^{2n}_2.
In Chapter 3 we apply an idea by H. Weber and P. Tsatsoulis employed in [TW16], to show tightness for the sequence of magnetization fluctuation fields of the Ising-Kac model on a periodic two-dimensional discrete torus near criticality and characterise the law of the limit as the Φ^4_2 measure on the torus. This result is not an immediate consequence of [MW17a].
In Chapter 4 we study the fluctuations of the magnetization field of the Ising-Kac model under the Kawasaki dynamic at criticality in a one dimensional discrete torus, and we provide some evidence towards the convergence in law to the solution to the Stochastic Cahn-Hilliard equation.

Chapter 1

Introduction

During the last few years there has been a huge development in the theory of SPDE, especially for what concerns the construction of solutions to ill-posed SPDE introduced in the physical literature in the last decades. The main source for the difficulties was the presence in the SPDE of both a nonlinearity and a rough noise term that forces the solution to be a distribution-valued process.

One of the first examples of such equations is given by the Stochastic Quantization Equation in dimension $d = 2$, also known as dynamical Φ^4_2 model, which can be formally described by the following SPDE, on $[0, T] \times \mathbb{T}^2$

$$\partial_t \Phi = \Delta\Phi - \Phi^3 + \xi \tag{1.1}$$

where $\Phi(0, \cdot) = \Phi_0(\cdot)$ is an initial condition and ξ a space-time white noise. When $d = 1$, the equation (1.1) is well posed and described by the theory of stochastic equations in infinite dimensions, presented for instance in [DPZ14]. For $d = 2$ a classical analysis shows that solutions to the dynamical Φ^4_2, are expected to be distributions, hence the cubic power in (1.1) makes the equation ill-posed.

In the attempt to solve the stochastic quantization equation in $d = 2$, one is tempted to consider the smooth approximation obtained solving (1.1) with a mollification of the noise

$$\partial_t \Phi^{(\epsilon)} = \Delta\Phi^{(\epsilon)} - (\Phi^{(\epsilon)})^3 + \xi^{(\epsilon)} .$$

As $\epsilon \to 0$ one expects the value $\Phi^{(\epsilon)}(t, x)$ to diverge for almost every point (t, x). This leads to the idea of *renormalization* of the equation via the introduction of a divergent quantity C_ϵ in such a way that the sequence of solutions of

$$\partial_t \Phi^{(\epsilon)} = \Delta\Phi^{(\epsilon)} - \left\{(\Phi^{(\epsilon)})^3 - C_\epsilon \Phi^{(\epsilon)}\right\} + \xi^{(\epsilon)}$$

converges to a non-trivial limit. Of course at this stage it is unclear if the process obtained in this way solves (1.1) in any way and how it depends on the sequence of the renormalizing constants.
In the study of the stochastic quantization equation Φ^4_2, a breakthrough was represented by the work [DPD03], where for the first time a pathwise solution has been constructed for

$$\partial_t \Phi = \Delta \Phi - \Phi^{:3:} + \xi \tag{1.2}$$

where $\Phi^{:3:}$ stands for the Wick power of Φ. This approach together with ideas from the theory of rough paths, ultimately led to the creation of a theory of regularity structures by M. Hairer [Hai14], that provides a general and abstract framework for the renormalization of such equations and the definition of a pathwise solution for the so-called subcritical SPDE. Very recently, some of the aforementioned works have been extended to the discrete setting [HM15, EH17], and therefore it appears natural to study discrete models arising from statistical mechanic.
This is motivated by the fact (see [JL91]) that the renormalization procedure appears to be encoded already within the framework of the microscopic dynamic, making the study of interacting particle systems the ideal environment for a discrete approach to ill-defined SPDE's.
Indeed, many of the examples of ill-posed SPDE's have been first introduced in the Physical literature starting from the analysis of discrete models (the most famous being the Ising model), and therefore it is natural to investigate whether or not the same renormalization can be performed at the level of the interacting particle system.

Another example is given by the KPZ equation, proposed by Kardar, Parisi and Zhang [KPZ86] as a description of the fluctuation of growing interfaces. For $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, the function $h(t, x)$ represents the fluctuation of the height of the interface at time t, the KPZ equation is formally given by

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi \tag{1.3}$$

where ξ is a space time white noise. Similarly to (1.1), equation (1.3) is ill-posed due to the presence of the partial derivative $\partial_x h$ and the lack of regularity of the solution (the solution of the stochastic heat equation, which is a linear approximation of (1.3), are almost $\frac{1}{2}$-Hölder in space [DPZ14]).

The main difficulty in the theory of particle systems however is its very same discrete structure. Indeed the price to pay is mainly represented by the loss of the Gaussian structure of the noise, which greatly simplifies the solution theory for the limiting ill posed differential equations.

One of the first result in this direction, has been obtained in [BPRS93, FR95] in the case of the Glauber dynamic in a one-dimensional Ising-Kac model at criticality, where the authors show the convergence in distribution of the fluctuation field to the solution of the stochastic quantization equation Φ^4_1. The idea of the proof is based on a coupling at the level of the spin system with a well studied discrete process, the voter model (see [Lig05, Chapter 5] for a description of the voter model).
In this case the solution of the limit process takes values in a space of functions, hence the limiting equation is already well posed and there is no need of renormalize microscopically the equation.
The first result where the renormalization plays a crucial role and can be embedded in the microscopic parameters of the model is [BG97], where the authors proved convergence of the weakly asymmetric simple exclusion process WASEP to the Cole-Hopf solution of the KPZ equation. In this result a nonlinear transform (the Gärtner transform) is used at the discrete level to rewrite the nonlinear problem into a linear one, which is then shown to converge to the solution of a martingale problem.
With a similar ideas, the authors of [MW17a] were able to prove that the fluctuation field for the Kac-Ising model at critical temperature converges to the solution of the stochastic quantization Φ^4_2. In order to do so, not only the microscopic model has to be rescaled with a diffusive scaling, but also the critical inverse temperature of the Kac-Ising model had to be tuned in a precise way, as a function of the lattice size. This discrepancy in dimension two was already known [CMP95, BZk97], and it played a crucial role in the renormalization of the nonlinear terms arising in the dynamic.
In a subsequent article, Shen and Weber [SW16] proved for a similar stochastic lattice model the convergence of the magnetization fluctuation field to the solution of the dynamical Φ^6_2 equation. The model they considered is the Kac-Blume-Capel model (or "site diluited" Ising model) under a Glauber-type dynamic around its critical temperature. Also in this case the parameters of the model (inverse temperature and chemical potential) need to be tuned, as a function of the Kac parameter, in a precise way and to converge to their respective critical values.

Looking at the aforementioned works it is possible to find some common features: both the discrete and continuous problems (that we call $\mathcal{P}^\gamma$ and $\mathcal{P}$, using the notation of [BPRS93]) have a *natural* linearized version (respectively $\mathcal{P}^\gamma_0$ and $\mathcal{P}_0$) associated to them. Moreover they satisfy

1. At the level of the limiting equation, there exists a non linear continuous map $\mathcal{S}$ that

couples (pathwise) the solution of the linear problem $\mathcal{P}_0$ with the solution of the limiting problem $\mathcal{P}$.

2. At the level of the particle system, the presence of a non linear map $\mathcal{S}_\gamma$, depending on the parameters of the model, that couples (pathwise) the solution of the linear discrete problem $\mathcal{P}_0^\gamma$ with the solution of the original version $\mathcal{P}^\gamma$ and such that, as $\gamma \to 0$, the map $\mathcal{S}_\gamma$ converges to $\mathcal{S}$ locally uniformly, i.e. for any compact set K

$$\lim_{\gamma\to 0} \sup_{z\in K} \|\mathcal{S}_\gamma(z) - \mathcal{S}(z)\| = 0$$

3. Convergence in distribution of the solution of the linearised discrete process $\mathcal{P}_0^\gamma$ to its continuous counterpart $\mathcal{P}_0$ (via a martingale formulation for instance).

We haven't been precise about the definition of the solution maps $\mathcal{S}_\gamma$ and $\mathcal{S}$ because they are going to depend on the precise definition of the models. Moreover in the second point we considered the uniform convergence on compact sets to keep the notations simple and because this is sufficient for the cases considered in Chapter 2. As we will remark in Chapter 4, the convergence in point 2) is not suitable for the case of the Kawasaki dynamic and has to be replaced by a convergence in probability, which seems to be a good tradeoff between the necessity of finding a deterministic discrete function $\mathcal{S}_\gamma$ of the linearised discrete problem $\mathcal{P}_0^\gamma$ and the and the need for $\mathcal{S}_\gamma$ to approximate the solution of the discrete problem.

1.1 Description of the Thesis

The present work is divided as follows: In the rest of Chapter 1 we are going to introduce the discrete model that we will be studying throughout the thesis, the Ising-Kac model, and present the particular Besov norms that we are going to use in the following chapters. In Chapter 2 we will prove that the magnetization fluctuation field of a generalization of the m-vector model on a two dimensional periodic torus, undergoing the Glauber dynamic, converges to a non linear system of SPDE's. To obtain the result it is necessary not only to control the precise rate of convergence of the inverse temperature to its critical value, but also the rate of convergence of a number of other parameters proportional to the degree of the nonlinearity in the limiting SPDE. The result represents a generalization of [MW17a, SW16] and it is obtained using essentially the same techniques. In Chapter 3 we will consider the sequence of the Gibbs measures of the Kac-Ising model on a periodic two dimensional lattice. We will prove that, as β approaches its critical value, the laws of the fluctuation of local magnetization field converge, as the lattice spacing vanishes, to the Φ^4_2 measure. The novelty of this theorem is not the result in itself, but the method of the proof. While the previous proof heavily relies on the correlation inequalities [CMP97] or on block spin approximation [SG73], the proof we are going to present is based on a dynamical approach. This approach doesn't rely crucially on the precise structure of the two-body potential nor on the combinatorial proprieties of the spin system and this suggests that it might be applied also to other models. In Chapter 4 we will investigate the fluctuations of the local magnetization field of the one dimensional Ising-Kac model evolving according to the Kawasaki dynamic at criticality. The treatment of the Kawasaki dynamic is essentially different from the approach in [MW17a] because the discrete evolution equation is not closed as a function of the fluctuation field. We will prove that closing the equation is sufficient to guarantee the convergence in law of the magnetization fluctuation field to the solution of the stochastic Cahn-Hilliard equation. We are not able to present a complete proof of the replacement lemma needed to obtain such result, however we will present an argument towards this replacement based on a version of the Boltzmann-Gibbs principle in [GJ13a] and on the one-block/two-blocks estimate [KOV89] tailored for the particular scaling used in Chapter 4.

1.2 Notations and definitions

We will now set some notations to be used in the following chapters. Throughout the thesis, the notations employed are mainly taken from [MW17a] and [BPRS93, FR95] for Chapters 2,3 and 4 and partly from [GJ14] for Chapter 4.

1.2.1 Statistical mechanics

Let $N \in \mathbb{N}$ be a positive integer and defined the periodic lattice $\Lambda_N^d \stackrel{\text{def}}{=} \{1-N,\dots,N\}^d$. Consider a state space S and let $\Sigma_N \stackrel{\text{def}}{=} S^{\Lambda_N^d}$ be the space of the configurations on Λ_N^d. In this thesis we will consider the state space S to be either $\mathbb{R}^m$ in Chapter 2 or $\{-1,1\}$ in Chapters 3 and 4. Given a configuration $\sigma \in \Sigma_N$, for $\Lambda' \subseteq \Lambda_N$ we denote with $\sigma_{\Lambda'}$ the configuration σ restricted on Λ' and for the singletons we simply write $\sigma_{\{x\}} = \sigma_x$ for $x \in \Lambda_N^d$.
We are now going to define the Ising-Kac model which will be the main object of investigation.

The Ising-Kac model

The Ising-Kac model is a spin system with ferromagnetic long range potential that has been introduced in statistical mechanics in [KUH63] for its simplicity and because it provides a framework to recover rigorously the van der Waals theory of phase transition.

Let $\mathfrak{K}$ be a rotation-invariant $\mathcal{C}^2(\mathbb{R}^2;[0,1])$ function with support contained in $B_0(3)$, the Euclidean ball of radius 3, satisfying

$$\int_{\mathbb{R}^2} \mathfrak{K}(x)\, dx = 1, \qquad \int_{\mathbb{R}^2} \mathfrak{K}(x)\, |x|^2\, dx = 4\ . \tag{1.4}$$

For $\gamma > 0$, define the *rescaled interaction kernel* $\kappa_\gamma : \Lambda_N^d \to [0,\infty)$ as

$$\kappa_\gamma(z) = \begin{cases} \bar{c}_\gamma\, \gamma^d \mathfrak{K}(\gamma|z|) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases} \tag{1.5}$$

where $\bar{c}_\gamma$ is a normalization constant such that $\sum_{z\in\Lambda_N\setminus\{0\}} \kappa_\gamma(z) = 1$. The choice $\kappa_\gamma(0) = 0$ in (1.5) has been made out of convenience and it is easily seen that the precise value of $kg(0)$ does not affect the definition of Gibbs measure.
Fix the state space $S = \{-1, 1\}$ and let $\sigma \in S^{\Lambda_N}$ be a configuration.
The Hamiltonian for the Ising-Kac model on the d-dimensional lattice with periodic boundary condition is defined by

$$\mathcal{H}_\beta^{\gamma,IK}(\sigma) = -\frac{\beta}{2} \sum_{x,y\in\Lambda_N^d} \kappa_\gamma(x-y)\sigma_x\sigma_y \tag{1.6}$$

where β is a non negative parameter which has the physical meaning of the inverse temperature.
On the space of configurations Σ_N, for any $b \in \mathbb{R}$, we can define the Gibbs measure as

$$\mu^N_{\gamma,\beta,b}(\sigma) \overset{\text{def}}{=} \big(\mathcal{Z}^N_{\gamma,\beta,b}\big)^{-1} \exp\Big\{ -\mathcal{H}_\beta^{\gamma,IK}(\sigma) + b \sum_{x\in\Lambda_N} \sigma_x \Big\} \tag{1.7}$$

where $\mathcal{Z}^N_{\gamma,\beta,b}$ is the normalizing factor that makes $\mu^N_{\gamma,\beta,b}$ a probability measure and it is called the partition function. The parameter b is commonly called *external magnetization* of the model.

Physically, the model describes the magnetic behavior of material made of atoms organized in a rigid structure, each of them having a spin. Each couple of spins has a tendency to be aligned and this is modeled by the measure (1.7) rewarding this behavior with a factor, inside the exponential, which is positive if both spins are aligned. This factor takes into account not only the distance between the spins, which depends on the parameter γ, but also the inverse temperature β. The definition of the Gibbs measure models the possibility

of placing the material inside an external magnetic field.
As the spin system describe the magnetic behavior of the material microscopically, one is interested in taking the limit $N \to \infty$ to find how the different parameters are influencing the macroscopic proprieties of the material. One of the macroscopic quantity that it is possible to observe is the internal magnetization

$$\mathcal{M}^N_{\gamma,\beta,b} \stackrel{\text{def}}{=} \mu^N_{\gamma,\beta,b}(\sigma_0)$$

which is the expected value of the spin at the origin (in a periodic lattice, the origin has no particular role among the sites). The limit $N \to \infty$ is called thermodynamic limit in the physical literature.
The partition function is an extremely useful quantity in the study of the Gibbs measure, since it encodes all the valuable information of the model as $N \to \infty$.
More precisely, the most useful pieces of information can be recover from the pressure per site

$$\psi^N_{\gamma,\beta,b} \stackrel{\text{def}}{=} |\Lambda_N|^{-1} \log \mathcal{Z}^N_{\gamma,\beta,b} \ .$$

For instance, the internal magnetization can be computed as the derivative of the pressure per site $\mathcal{M}^N_{\gamma,\beta,b} = \partial_b \psi^N_{\gamma,\beta,b}$.

The role of the Gibbs measure is strictly connected with the concept of phase transitions that investigate the dependency of the macroscopic observables (the internal magnetization for instance) from the boundary conditions of each set Λ_N, namely the values of the spins outside the domain Λ_N. For ferromagnetic systems, the dependence on the boundary conditions grows with the value of the inverse temperature β. In a very general setting, for small values of β there is no phase transition [Lig05, Chapter 4, Theorem 3.1], hence contribution of the boundary conditions is forgotten as $N \to \infty$. In two dimension a Peierls argument is usually sufficient to guarantee the presence of phase transition for β large enough, see for instance [Gri64]. Therefore there exists a critical value β_c such that for $\beta > \beta_c$ the boundary condition casts a non trivial influence over the macroscopic quantities of the system. For the Ising model in two dimension, the value of the critical temperature has been successfully computed by L. Onsager (see [Bax89, Chapter 7]). Since we always consider the Hamiltonian (1.6) with periodic boundary conditions, our definition of the Gibbs measure does not fall in the classical framework and in this thesis we will not discuss the problem of phase transition. Nonetheless we need to justify the notion of “criticality” in the title of this thesis and the particular value of $\beta_c = 1$ considered in the next chapters. The Kac-Ising model gained popularity from the result [LP66], where the authors proved that the limit of the pressure

$$\lim_{\gamma\to 0}\lim_{N\to\infty}\psi^N_{\gamma,\beta,b}$$

exists and coincide with the infinite volume pressure of the mean field model (Curie-Weiss model) with parameters β, b (see [Pre09, Theorem 4.2.1.1] for more details). As a consequence all the macroscopic thermodynamic proprieties coincide, after the disjoint limit, with the proprieties of the Curie-Weiss model, including the critical value of the inverse temperature. The critical value of the inverse temperature for the Curie-Weiss model coincides with the value of β that separates the behaviour of the equation

$$x = \tanh(\beta x)$$

in terms of number of real solutions [Bax89, Chapter 3] and it is given by $\beta_c = 1$ in any dimension.
The limit (in order) $\lim_{\gamma\to 0}\lim_{N\to\infty}$ is often called the Lebowitz-Penrose limit. As a difference from the Lebowitz-Penrose limit, we ask γ and ϵ to satisfy a precise relation and we will fix the inverse temperature to be a precise function of the range of the potential $\beta = \beta(\gamma)$ that satisfies $\lim_{\gamma\to 0}\beta(\gamma) = \beta_c = 1$ (see 2.20).
The Ising-Kac model has already been proven useful to study the Φ^4_d theory, see [GK85], where a renormalisation group approach has been used to approximate Φ^4_d with generalized Ising models, and [SG73] with classical Ising spins.

1.2.2 Discrete and continuous Besov spaces

We will think the discrete lattice Λ^d_N to parametrize a discretization of the d-dimensional torus $\mathbb{T}^d$ that we identify with $[-1,1]^d$ with periodic boundary conditions. For this purpose let $\epsilon = N^{-1}$ and $\Lambda^d_\varepsilon \stackrel{\text{def}}{=} \epsilon\Lambda^d_N \subseteq \mathbb{T}^d$ the discretisation induced on $\mathbb{T}^d$. All the summations over $\Lambda^d_N, \Lambda^d_\varepsilon, \mathbb{T}^d$ are understood with periodic boundary conditions. For $f, g : \Lambda^d_\varepsilon \to \mathbb{C}$ we will use the following notations

$$\|f\|^p_{L^p(\Lambda^d_\varepsilon)} \stackrel{\text{def}}{=} \sum_{x\in\Lambda^d_\varepsilon} \epsilon^d |f(x)|^p \,, \qquad \langle f, g\rangle_{\Lambda^d_\varepsilon} \stackrel{\text{def}}{=} \sum_{x\in\Lambda^d_\varepsilon} \epsilon^d f(x)\overline{g(x)}$$

respectively for the discrete L^p norm and the scalar product.
We will denote also the discrete convolution as

$$f *_\epsilon g(x) \stackrel{\text{def}}{=} \sum_{y\in\Lambda^d_\varepsilon} \epsilon^d f(x-y)g(y) \,, \qquad \text{for } x \in \Lambda^d_\varepsilon$$

and we will make an extensive use of the Fourier transform

$$\widehat{f}(\omega) \stackrel{\text{def}}{=} \sum_{x\in\Lambda_\varepsilon^d} \epsilon^d f(x) e^{-\pi i\omega\cdot x} \qquad \text{for } \omega \in \Lambda_N^d\,.$$

The Fourier inversion theorem with this notation reads

$$f(x) = \frac{1}{2^d} \sum_{\omega\in\Lambda_N^d} \widehat{f}(\omega) e^{\pi i w\cdot x} \qquad \text{for } x \in \Lambda_\varepsilon^d\,. \tag{1.8}$$

We shall use the same notation $\mathsf{Ext}(f)$ as in [MW17a] to denote the extension of f to the continuous torus $\mathbb{T}^d$ via (1.8) applied to $x \in \mathbb{T}^d$, namely

$$\mathsf{Ext}(f)(x) \stackrel{\text{def}}{=} \frac{1}{2^d} \sum_{\omega\in\Lambda_N^d} \widehat{f}(\omega) e^{\pi i w\cdot x} \qquad \text{for } x \in \mathbb{T}^d\,. \tag{1.9}$$

We will measure the regularity of a function $g : \mathbb{T}^d \to \mathbb{R}$ with the Besov norm, defined for $\nu \in \mathbb{R}$, and $p, q \in [1, \infty]$ as

$$\|g\|_{\mathcal{B}^\nu_{p,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k\geq -1} 2^{\nu k q} \|\delta_k g\|^q_{L^p(\mathbb{T}^d)}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k\geq -1} 2^{\nu k} \|\delta_k g\|_{L^p(\mathbb{T}^d)} & \text{if } q = \infty \end{cases} \tag{1.10}$$

where δ_k denotes the Paley-Littlewood projection defined in (A.1).
Following [MW17a], we will define the Besov spaces $\mathcal{B}^\nu_{p,q}(\mathbb{T}^d)$ as the completion of the set of smooth test functions over the torus equipped with the Besov norm

$$\mathcal{B}^\nu_{p,q}(\mathbb{T}^d) \stackrel{\text{def}}{=} \overline{\mathcal{S}(\mathbb{T}^d)}^{\mathcal{B}^\nu_{p,q}} \tag{1.11}$$

In particular, the parameter $\nu \in \mathbb{R}$ represents the regularity of the function and the space $\mathcal{B}^\nu_{p,q}(\mathbb{T}^d)$ contains distributions if $\nu < 0$.
When $p = q = \infty$, we shall denote with $\mathcal{C}^\nu(\mathbb{T}^d)$ the Besov space $\mathcal{B}^\nu_{\infty,\infty}(\mathbb{T}^d)$.

Remark 1.2.1 It is important to remark that when $p = q = \infty$, the above definition of Besov space does not coincide with the usual definition in the literature, for example in [BCD11, Chapter 2].
The main advantage of the definition (1.11), is that $\mathcal{C}^\nu(\mathbb{T}^d)$ is automatically a separable space. This is important for instance in the proof of Theorem 3.2.4.

We are going to use the Besov norm to measure the regularity of functions defined on the lattice $g : \Lambda_\varepsilon^d \to \mathbb{R}$. There are two possible ways to extend the definition of Besov norm in the discrete setting and we are going to use both of them in this thesis:

- The most natural among the definitions is obtained extending the function $g : \Lambda_\varepsilon^d \to \mathbb{R}$ to the continuous torus $\mathbb{T}^d$ via the operator Ext defined in (1.9) and

$$\|g\|_{\mathcal{B}^\nu_{p,q}} \stackrel{\text{def}}{=} \|\mathsf{Ext}(g)\|_{\mathcal{B}^\nu_{p,q}}$$

  and in this way it is possible to make use of the proprieties proven in the literature for continuous Besov spaces.

- Another possible way is to use the discrete $L^p(\Lambda_\varepsilon^d)$ norm and discrete convolution instead of $L^p(\mathbb{T}^d)$ in (1.11)

$$\|g\|_{\mathcal{B}^\nu_{p,q}(\Lambda_\varepsilon^d)} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k\geq -1} 2^{\nu k q} \|\delta_k \mathsf{Ext}(g)\|^q_{L^p(\Lambda_\varepsilon^d)}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k\geq -1} 2^{\nu k} \|\delta_k \mathsf{Ext}(g)\|_{L^p(\Lambda_\varepsilon^d)} & \text{if } q = \infty \end{cases} \tag{1.12}$$

  This discrete version of the Besov norm will only be used in Chapter 3 and it doesn't seem to be present in the literature, up to our knowledge.

As it is easy to see from the definitions, we always have that

$$\|g\|_{\mathcal{B}^\nu_{\infty,q}(\Lambda_\varepsilon^d)} \leq \|g\|_{\mathcal{B}^\nu_{\infty,q}(\mathbb{T}^d)} \ .$$

Furthermore, one could prove that, for general $p \geq 1$, the discrete Besov norm is controlled by its continuous version, as it is shown in Proposition A.0.7

$$\|g\|_{\mathcal{B}^\nu_{p,q}(\Lambda_\varepsilon^d)} \leq C(\nu, p, q) \|g\|_{\mathcal{B}^\nu_{p,q}(\mathbb{T}^d)} \ .$$

At this point the reader could ask about the necessity of the introduction of two similar norms. The concrete motivation lies in a technical inequality in Chapter 3, proven in the Appendix A, Lemma A.0.11 for the discrete Besov norm, but not for the continuous one. Proving Lemma A.0.11 with the continuous Besov norm instead, would make the introduction of the discrete norm obsolete, at least for the present work.

Remark 1.2.2 It is immediate to see that (1.12) defines a norm on the space of functions $\mathbb{R}^{\Lambda_\varepsilon^d}$ because of the equality

$$g(x) = \mathsf{Ext}(g)(x) = \sum_{k\geq -1} \delta_k \mathsf{Ext}(g)(x) \qquad \text{for all } x \in \Lambda_\varepsilon^d$$

valid for all the gridpoints.

As in [DPD03], the main motivation behind the use of the Besov norm is the following crucial proposition, which is proven in Appendix A in the case of the discrete Besov spaces.

Proposition 1.2.3 (Multiplicative inequality) *Let $a, b > 0$ with $a < b$. Assume f to be in $\mathcal{C}^{-a}(\mathbb{T}^d)$ and g to be in $\mathcal{C}^b(\mathbb{T}^d)$. Then the pointwise product fg (well defined on a dense subspace of $\mathcal{C}^{-a}(\mathbb{T}^d)$) can be extended to a bilinear continuous map $\mathcal{C}^{-a}(\mathbb{T}^d) \times \mathcal{C}^b(\mathbb{T}^d) \to \mathcal{C}^{-a}(\mathbb{T}^d)$ and*

$$\|fg\|_{\mathcal{C}^{-a}(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{C}^{-a}(\mathbb{T}^d)} \|g\|_{\mathcal{C}^b(\mathbb{T}^d)} \ . \tag{1.13}$$

For a more detailed definition and proprieties of the above Besov norm, we refer to Appendix A, which also contains the proofs of inequalities involving the discrete Besov norm (1.12).

Chapter 2

Convergence of Glauber dynamic of Ising-like models to Φ_2^{2n}

2.1 Introduction

The aim of this chapter is to prove the robustness of the method employed by J.C. Mourrat and H. Weber in [MW17a] applying the same techniques to a generalized version of the Ising-Kac model. The study is motivated by a conjecture in [SW16], where the authors extended the work of [MW17a] to the so called Kac-Blume-Capel model, undergoing a Glauber dynamic.

Recall the definitions given in Section 1.2.1. The Blume-Capel model, also known a site-diluted model, is a modification of the classic Ising model, that takes into account the possibility of having empty sites in the lattice, which do not interact magnetically with the spins in the other sites.

In this chapter we will assume the notations of Chapter 1, with one difference: since we will be working in dimension 2, we will omit the d in Λ_N^d and Λ_ε^d and we will use $\Lambda_N = \{-N+1, N\}^2$ and $\Lambda_\varepsilon = (\epsilon\mathbb{Z})^2 \cap (0,1]^2$. In the Kac-Blume-Capel each spin is allowed to take value in the set $S = \{-1, 0, 1\}$. More precisely, for a configuration $\sigma \in \{-1,0,1\}^{\Lambda_N}$, the Hamiltonian of the Kac-Blume-Capel model is given by

$$\mathcal{H}_{\beta,\theta}^{\gamma,KBC}(\sigma) = -\frac{\beta}{2} \sum_{x,y\in\Lambda_N^2} \kappa_\gamma(x-y)\sigma_x\sigma_y - \theta \sum_{x\in\Lambda_N^2} \sigma_x^2 \,. \tag{2.1}$$

The parameter θ in last term in (2.1), has the role of a chemical potential, adjusting the density of the magnetic spins in the lattice. For $\theta \to \infty$, the Gibbs measure favors configuration with a non zero spin in every lattice, obtaining the Ising-Kac model described in Section 1.2.1 as a special case.

Using the definition of the Hamiltonian (2.1) it is possible to define the Glauber dynamics (spin flip dynamics) on the configuration space of the Kac-Blume-Capel model. In [SW16, Theorem 2.5] the authors showed that, for specific values of the inverse temperature and the chemical potential, depending on γ, suitably rescaling the space and the time, the fluctuation field of the local magnetization associated to the Kac-Blume-Capel model, converges in distribution to the solution of the stochastic quantization equation Φ^6_2

$$dX = \left(\Delta X - \frac{9}{20} X^{:5:}\right) dt + \sqrt{2/3}\, dW \,.$$

In [SW16] (see discussion after the Meta-theorem 1.1), the authors conjectured, for all $n > 1$, the existence of a spin system, such that the fluctuation of the magnetization field under the Glauber dynamic converges to the solution of the stochastic quantization equation Φ^{2n}_2

$$dX = \left(\Delta X - X^{:2n-1:}\right) dt + \sqrt{2} dW \,. \tag{2.2}$$

We recall that it is nontrivial to interpret equation (2.2) and its notion of solution. In [DPD03] the authors showed the existence and uniqueness of strong solutions of (2.2) for any initial condition in a Besov space of negative regularity.

In order to prove the convergence to (2.2) one has the feeling that it would be necessary to provide the model with enough parameters, in addition to the "scaling" parameter, each of them converging to a "critical" value in a precise way as $\gamma \to 0$. They would play the same role of the chemical potential in the Kac-Blume-Capel model. One of the reason for the introduction of so many parameters is that all the monomials in (2.2) need to be renormalized in dimension two. It turns out that it is possible to do so indirectly adding a one-body potential with all the "model" parameters.

Consider an odd polynomial $\mathfrak{a}_1 + \mathfrak{a}_2 x + \cdots + \mathfrak{a}_{2n-1} x^{2n-1}$ with negative leading coefficient and let $m \geq 1$ be a positive integer.
In the present chapter we extend the results of [MW17a, SW16] and prove the above conjecture. More precisely, we describe how to produce a (vector-valued) spin systems on a periodic lattice together with a Gibbs measure and a Glauber dynamic on it, such that its fluctuation field converges in distribution to the solution to the following SPDE

$$dX = \left(\Delta X + \mathfrak{a}_1 X + \mathfrak{a}_3 : X|X|^2 : + \cdots + \mathfrak{a}_{2n-1} : X|X|^{2n-2} :\right) dt + \sqrt{2} dW \tag{2.3}$$

where $X = (X^{(1)}, \ldots, X^{(m)})$ is a vector-valued distribution from the 2-dimensional torus and $W = (W^{(1)}, \ldots, W^{(m)})$ is a collection of m independent space-time Wiener process on $L^2([0,T];\mathbb{T}^2)$. In the above equation $:p(X):$ denotes the Wick renormalization of the polynomial p and $|X| = \left(X^{(1)2} + \cdots + X^{(m)2}\right)^{\frac{1}{2}}$ is the Euclidean norm of the vector and it is always present in (2.3) squared. The negative leading coefficient is necessary to guarantee the existence of the solution to (2.3) for all times.

The framework we are going to describe not only covers the results in [MW17a, SW16] as a particular case, but also applies to other models of statistical mechanics with Hamiltonian similar to (2.1).
One of them is the m-vector model, a spin system with state space $S = \{|v| = 1 : v \in \mathbb{R}^m\}$ and Hamiltonian

$$\mathcal{H}_\beta^{\gamma,VEC(m)}(\sigma) = -\frac{\beta}{2} \sum_{x,y\in\Lambda_N^2} \kappa_\gamma(x-y) \langle \sigma_x, \sigma_y \rangle \qquad \text{for } \sigma \in S^{\Lambda_N^2}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^m$.
As an application of our main result, we deduce in Corollary 2.2.22 that the fluctuation field of the Glauber dynamic on the m-vector model converges in distribution to the solution of the system of SPDE

$$dX = \left(\Delta X - \frac{m}{m+2} : |X|^2 X :\right) dt + \frac{1}{\sqrt{m}} dW \tag{2.4}$$

The main difference technical differences between [MW17a] and the content of this chapter is coming from the fact that we do not want to assume the state space S to be bounded. This allows, for instance, to consider models $\{\Phi_x\}_{x\in\Lambda_N^2}$ continuous spin state with Gibbs measure similar to

$$\exp\left\{\frac{\beta}{2} \sum_{x,y\in\Lambda_N^2} \kappa_\gamma(x,y) \langle X_x, X_y \rangle - \sum_{x\in\Lambda_N^2} |X_x|^\vartheta \right\}$$

for $\vartheta > 1$. The idea behind it is the observation that the precise form of the polynomial in (2.3) depends on the state space of the system and on the one-body potential of the specification of the Gibbs measure, while the two-body potential is constrained to a precise form. It is natural therefore to allow the state space and the one-body potential to be as general as possible: we shall condensate the contribution coming from the one-body potential into an "a priori" reference measure, that we shall call ν_γ. The a priori measure will depend on the model parameters, that are in turn depending on γ, the main parameter which is orchestrating the convergence of the model.

In Section 2.2 we will introduce the models and the reference measure on the state space of the spins and we define the dynamic. In Proposition 2.2.14 we prove that, under some assumption over $\mathfrak{a}_{2n-1}$, there exists a (discrete) model realizing (2.3). In Section 2.2.6 we recall the solution theory of (2.3) and we introduce some ingredients for the following sections. In Sections 2.3 and 2.4 the linear part of the process is shown to converge to the solution of the stochastic heat equation. Section 2.5, completes the analysis with the study of the nonlinear part of the dynamic. The contenent of this chapter has been made public in [Ibe17].

2.2 Models and main theorem

Let $m \geq 1$ be a positive integer and let $S \subseteq \mathbb{R}^m$ be the state space for the spins. We will consider a set of isotropic reference measures $\{\nu_\gamma\}_\gamma$ on S, normalized to have $\int_S |\eta|^2 \nu_\gamma(d\eta) = m$ and satisfying, for all $\theta > 0$

$$\int_S e^{\theta|\eta|} \nu_\gamma(d\eta) < \infty$$

uniformly in γ.
The last request is trivially satisfied if the state space is compact but takes into account the possibility to have unbounded spins (for instance Gaussian) and contains the framework of the previous works [MW17a, SW16].
In addition to the above requirements, ν_γ will have to satisfy constraints related to the form of the limiting polynomial. In order to understand the form of the further assumptions, it is necessary to introduce the model and the dynamic. In the following pages we are going to define the dynamic and in Subsection 2.2.3 we will complete the list of assumptions on ν_γ.

On this set, we define a product measure $\nu_\gamma^N \stackrel{\text{def}}{=} \prod_{i \in \Lambda_N} \nu_\gamma^{(i)}$, where each $\nu_\gamma^{(i)}$ is a copy of ν_γ at the position i in the lattice. The Gibbs measure will be defined by prescribing its density with respect to ν_γ^N which we call the *reference product measure*.

Remark 2.2.1 It seems strange to allow the measure ν_γ to depend on γ. The reason is that in order to obtain a generic polynomial as in (2.3), we need the moments of the a priory measure ν_γ to satisfy some relations as γ tends to 0. In [SW16], choosing the model parameters $(\beta, \theta) = (\beta(\gamma), \theta(\gamma))$ close to a critical curve, it is shown that the Glauber dynamic converges to the solution of the dynamical stochastic quantization equation Φ^4_2, while for $(\beta(\gamma), \theta(\gamma))$ close to a critical point, one obtains the convergence to Φ^6_2. The reason is basically that some algebraic relations among the parameters have to be satisfied in order to annihilate more coefficients. This is not the only constraint that the model parameters have to satisfy: since the solutions of the limiting equation are distribution valued processes, the divergences created by the powers of the variables have to be compensated tuning the direction and the speed at which the model parameters approach the critical hypersurface. We remark furthermore that the parameter $\beta(\gamma)$ itself could have been absorbed into the measure ν_γ. In fact we requested $\int_S |\eta|^2 \nu_\gamma(d\eta) = m$ to allow clear definition for the model and for the inverse temperature as well.

The set $(\beta(\gamma), \nu_\gamma)$ will be referred to as the set of *model parameter*, to distinguish them from the parameter involved in the space time rescaling of the Markov chain.
In order to keep the notation light, we will drop from $\beta(\gamma)$ the dependence on γ and assume β to be always dependent on γ.

We will now going to define the Gibbs measure for the spin system. Recall the form of the potential κ_γ introduced in Section 1.2.1. For $\sigma \in \Sigma_N$ define

$$\mathcal{H}_\beta^\gamma(\sigma) = -\frac{\beta}{2} \sum_{x,y \in \Lambda_N} \kappa_\gamma(x-y) \, \langle \sigma_x, \sigma_y \rangle \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^m$. The Gibbs measure μ_γ^N on Σ_N is defined, using the reference measure introduced above, as

$$\mu_\gamma^N(\sigma) \stackrel{\text{def}}{=} \left(\mathcal{Z}_\gamma^N\right)^{-1} \exp\Big\{ - \mathcal{H}_\beta^\gamma(\sigma) \Big\} \nu_\gamma^N(\sigma) \tag{2.6}$$

where $\mathcal{Z}_\gamma^N$ is the normalization constant. We define, for $x \in \Lambda_N$ the *local magnetization field*

$$h_\gamma(x, \sigma) \stackrel{\text{def}}{=} \sum_{i \in \Lambda_N} \kappa_\gamma(x-i) \sigma_i \, . \tag{2.7}$$

And we will soon abuse the notation writing $h_\gamma(x,t)$ instead of $h_\gamma(x, \sigma_t)$, and $h_\gamma(x)$ for $h_\gamma(x,\sigma)$ when there is no time involved.
We will now define a dynamic over the state space Σ_N as follows: each site of the discrete lattice is given an independent Poisson clock with rate 1. When the clock rings at site $x \in \Lambda_N$, the spin at x which is in the state σ_x, changes its value by picking one randomly distributed according to a distribution which makes the Gibbs measure (2.6) reversible for the Markov process described. When a jump occurs at $x \in \Lambda_N$, the configuration changes

$$\sigma_{\Lambda_N \setminus \{x\}} \sqcup \sigma_x \mapsto \sigma_{\Lambda_N \setminus \{x\}} \sqcup \eta_x$$

and the new value η_x is chosen according a probability distribution on $\mathbb{R}^m$ depending on the energy difference between the configuration before and after the jump

$$\Delta^x \mathcal{H}_\beta^\gamma \stackrel{\text{def}}{=} \mathcal{H}_\beta^\gamma(\sigma_{\Lambda_N \setminus \{x\}} \sqcup \eta_x) - \mathcal{H}_\beta^\gamma(\sigma) = \langle \beta h_\gamma(x), (\eta_x - \sigma_x) \rangle \ ,$$

where we denoted by $\sqcup$ the operation that concatenates two configurations defined for any subset $A \subseteq \Lambda_N$ by

$$\left(\sigma_{\Lambda_N \setminus A} \sqcup \eta_A\right)_y = \begin{cases} \eta_y & \text{if } y \in A \\ \sigma_y & \text{if } y \notin A \end{cases} .$$

We are now going to introduce a "tilted" version of the measure ν_γ which will be the distribution of the spin at x after a jump at x has occurred. For a vector $\lambda \in \mathbb{R}^m$ we define p^λ as the measure on S prescribing its density with respect to ν_γ proportional to

$$p^\lambda(\eta_x) \sim \exp(\beta \langle \lambda, \eta_x \rangle) \nu_\gamma(\eta_x) \ . \tag{2.8}$$

When a jump occurs at site $x \in \Lambda_N$, we choose the next value of the spin according to $p^{h_\gamma(x)}(d\eta_x)$. This is the choice of rate function for Glauber dynamic.

Remark 2.2.2 The distribution $p^{h_\gamma(x)}$ only depends on σ via $h_\gamma(x) = \sum_{i \neq x} \kappa_\gamma(x-i)\sigma_i$ and does not depend on the new value of σ_x. This guarantees the predictability of the rate function.

Remark 2.2.3 It turns out that it will be more convenient to work with a modified version of $p^{h_\gamma(x)}$, that will be introduced in Subsection 2.2.5. The process considered this way, will coincide with the Glauber dynamic defined here "up to a stopping time" as in [MW17a].

It will be convenient to introduce the following

$$\Phi(\lambda) \stackrel{\text{def}}{=} \int_S \eta_x p^\lambda(d\eta_x) = \frac{\int_S \eta_x e^{\beta\langle \lambda, \eta_x \rangle} d\nu_\gamma(d\eta_x)}{\int_S e^{\beta \langle \lambda, \eta_x \rangle} d\nu_\gamma(d\eta_x)} \ , \tag{2.9}$$

which is the mean value of the the distribution of the next spin after a jump.
The Glauber dynamic is described by the following generator: for every $f : \Sigma_N \to \mathbb{R}$ and $\sigma \in \Sigma_N$

$$\mathcal{L}_\gamma^G f(\sigma) = \sum_{x \in \Lambda_N} \int_S p^{h_\gamma(x)}(d\eta_x) \left(f(\eta_x \sqcup \sigma_{\{x\}^c}) - f(\sigma) \right) \ .$$

If $f(\sigma) = h_\gamma(z, \sigma)$ then

$$\begin{aligned} \mathcal{L}_\gamma^G h_\gamma(z) &= \sum_{j \neq z} \kappa_\gamma(j,z) \mathcal{L}_\gamma^G \sigma_j = \sum_{j \neq z} \kappa_\gamma(j,z) \int_S \eta_j p^{h_\gamma(j)}(d\eta_j) - h_\gamma(z) \\ &= \kappa_\gamma * \Phi(h_\gamma(\cdot))(z) - h_\gamma(z), \end{aligned} \tag{2.10}$$

where we used the fact that $\kappa_\gamma(0) = 0$.

Our study will develop around the evolution in time of the local mean field $h_\gamma(x,t)$. Here (x,t) are "microscopic coordinates" and we have

$$h_\gamma(x,t) = \int_0^t \mathcal{L}_\gamma^G h_\gamma(x,s)ds + m_\gamma(x,t), \tag{2.11}$$

where m_γ is an $\mathbb{R}^m$-valued martingale with predictable quadratic variation given by the matrix, for $1 \le i,j \le m$

$$\begin{aligned}&\left\langle m_\gamma^{(i)}(x,\cdot), m_\gamma^{(j)}(z,\cdot)\right\rangle_t \\ &= \int_0^t \sum_{l\in\Lambda_N} \kappa_\gamma(l-x)\kappa_\gamma(l-z)\int_S (\eta^{(i)} - \sigma_l^{(i)}(s^-))(\eta^{(j)} - \sigma_l^{(j)}(s^-))p^{h_\gamma(x,s^-)}(d\eta)\,.\end{aligned} \tag{2.12}$$

Proposition 2.2.4 *The Gibbs measure μ_γ^N is reversible with respect to the above dynamic.*

Proof. In order to do this we are using the fact that $h_\gamma(x)$ remains unchanged by the jump at x, by the definition (2.7) and (1.5), hence the measure $p^{h_\gamma(x)}(d\eta_x)$ as well. We have

$$\begin{aligned}&\int f(\sigma)\mathcal{L}_\gamma^G g(\sigma)\mu_\gamma^N(d\sigma) \\ &= \int f(\sigma)\left(\sum_{x\in\Lambda_N}\int_{\mathbb{R}^m} p^{h_\gamma(x)}(d\eta_x)g(\eta_x \sqcup \sigma_{\{x\}^c}) - g(\sigma)\right)\mu_\gamma^N(\sigma) \\ &= \sum_{x\in\Lambda_N}\int f(\sigma)g(\eta_x\sqcup\sigma_{x^c})\frac{d\mu_\gamma^N}{d\nu_\gamma^N}(\sigma)\nu_\gamma^N(d\sigma)\frac{dp^{h_\gamma(x)}}{d\nu_\gamma}(\eta_x)\nu_\gamma(d\eta_x) - \int f(\sigma)g(\sigma)\mu_\gamma^N(d\sigma)\end{aligned}$$

where we used the fact that $\frac{dp}{d\nu_\gamma}$ is defined in (2.8). Since $p^{h_\gamma(x)}$ doesn't contain the variable σ_x and

$$\frac{d\mu_\gamma^N}{\nu_\gamma^N}(\sigma)\frac{dp^{h_\gamma(x)}}{d\nu_\gamma}(\eta_x) = \frac{d\mu_\gamma^N}{\nu_\gamma^N}(\eta_x\sqcup\sigma_{\{x\}^c})\frac{dp^{h_\gamma(x)}}{d\nu_\gamma}(\sigma_x) \qquad \nu_\gamma^N\otimes\nu_\gamma - a.s.\,,$$

and we conclude that μ_γ is reversible with respect to the generator $\mathcal{L}_\gamma^G$. □

Remark 2.2.5 We would like to point out that this framework covers the cases in [MW17a] and [SW16], having respectively

$$\text{(Kac-Ising)} \qquad \nu_\gamma^I = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$
$$\text{(Kac-Blume-Capel)} \qquad \nu_\gamma^{BC} = \frac{e^\theta}{1+2e^\theta}\delta_{-1} + \frac{1}{1+2e^\theta}\delta_0 + \frac{e^\theta}{1+2e^\theta}\delta_1$$

where δ_x is the Dirac measure at x. Here $m = 1$ (recall that m is the dimension of the state space). Moreover the rate function for the Kac-Blume-Capel model (formula 2.5 in [SW16]), can be written using (2.8)

$$p^{h_\gamma(x)}(d\eta_x) = \frac{\exp\left(\beta h_\gamma(x)\eta_x\right)\nu_\gamma^{BC}(d\eta_x)}{\int_{\mathbb{R}}\exp\{\beta h_\gamma(x)\zeta\}\nu_\gamma^{BC}(d\zeta)} = \begin{cases} \frac{e^{-\beta h_\gamma(x)+\theta}}{1+e^{-\beta h_\gamma(x)+\theta}+e^{\beta h_\gamma(x)+\theta}} & \text{if } \eta_x = -1 \\ \frac{1}{1+e^{-\beta h_\gamma(x)+\theta}+e^{\beta h_\gamma(x)+\theta}} & \text{if } \eta_x = 0 \\ \frac{e^{\beta h_\gamma(x)+\theta}}{1+e^{-\beta h_\gamma(x)+\theta}+e^{\beta h_\gamma(x)+\theta}} & \text{if } \eta_x = 1 \end{cases} .$$

We observe that

$$\int_{\mathbb{R}} x^2 \nu_\gamma^{BC}(dx) = \frac{2e^\theta}{1+2e^\theta} \neq m \ , \tag{2.13}$$

hence ν_γ^{BC} doesn't satisfy the third assumption for a reference measure (this is not a problem as it is always possible to rescale the value of spins). This minor difference is responsible for the fact that the critical temperature of the Kac-Blume-Capel model considered in [SW16] is different from the critical temperatures in our models. In a similar way, if $m > 1$ the m-vector model is a generalization of the Ising model defined for

$$\text{(m-vector model)} \qquad \nu_\gamma(\eta) = \Omega_m^{-1}\delta_1(|\eta|)$$

where Ω_m is the surface of the $m-1$-dimensional sphere. Also in this case $\int_S |\eta|^2\nu_\gamma(d\eta) = 1 \neq m$.

2.2.1 Rescaling

We are interested in the fluctuation of the local magnetization h_γ when the parameter β is close to its mean field critical value β_c. In order for the fluctuations to survive the rescaling process, we must ensure that the quantity $\mathcal{L}_\gamma^G h_\gamma$ doesn't dominate the fluctuations. We now present some heuristic calculations to justify the choice of the scaling that we will choose. Separate the generator into its diffusive part and the nonlinear part

$$\mathcal{L}_\gamma^G h_\gamma(z) = \kappa_\gamma * \left(\Phi(h_\gamma(\cdot)) - h_\gamma\right)(z) + \kappa_\gamma * h_\gamma(z) - h_\gamma(z)$$

then rewrite (2.9) as

$$
\begin{aligned}
&\Phi(h_\gamma(\cdot))(z) - h_\gamma(z) \\
&\qquad\qquad = \nabla_\lambda|_{\lambda=\beta h_\gamma(z)} \left(\log \int_S e^{\langle \lambda, \eta\rangle} \nu_\gamma(d\eta) - \frac{|\lambda|^2}{2\beta} \right) \\
&\qquad\qquad\qquad = \nabla_\lambda|_{\lambda=\beta h_\gamma(z)} \log \int_S e^{\langle \lambda, \eta\rangle - \frac{|\lambda|^2}{2\beta}} \nu_\gamma(d\eta) \,. \quad (2.14)
\end{aligned}
$$

At this point one recognizes the fact that, if ν_γ were a m-dimensional Gaussian with mean zero and covariance matrix $\beta^{-1}I$, the quantity would vanish. In order to make some of the powers of λ vanish, we would need ν_γ to have some moments in common with a Gaussian measure, when $\gamma \to 0$. The more moments of ν_γ are Gaussian, the higher the degree of the polynomial. This means that only a fixed number of moments of ν_γ will be involved in the production of the limit polynomial in (2.3). This will be further explained in Subsection 2.2.4.

In order to make $h_\gamma(z)$ the Taylor expansion at fist order of $\Phi(h_\gamma(\cdot))(z)$, the value of $\beta = \beta(\gamma)$ will be taken suitably close to 1, and its precise value will be given in (2.37). This discrepancy will be used to compensate the divergences in the renormalization of the of the field X_γ.

From the fact that ν is rotation invariant, it follows that $\Phi(h_\gamma(\cdot))(z)$ is a vector with the same direction of h_γ. Therefore $\Phi(h_\gamma(\cdot))(z)$ coincides with its projection to the vector $h_\gamma(z)$ which is given by

$$
\Phi(h_\gamma(\cdot))(z) = \left(\frac{\int_S \langle h_\gamma, \eta\rangle \, e^{\langle \beta h_\gamma(z), \eta\rangle} \nu_\gamma(d\eta)}{\int_S e^{\langle \beta h_\gamma(z), \eta\rangle} \nu_\gamma(d\eta)} \right) \frac{h_\gamma(z)}{|h_\gamma(z)|^2}
$$

and a Taylor expansion of the exponential yields

$$
\begin{aligned}
\Phi(h_\gamma(\cdot))(z) - h_\gamma(z) &= \frac{h_\gamma(z)}{|h_\gamma(z)|^2} \left(\frac{\sum_{j \text{ odd}} \frac{1}{j!} \beta^j \int |\, \langle h_\gamma(z), \eta\rangle \,|^{j+1} \nu_\gamma(d\eta)}{\sum_{j \text{ even}} \frac{1}{j!} \beta^j \int |\, \langle h_\gamma(z), \eta\rangle \,|^{j} \nu_\gamma(d\eta)} \right) - h_\gamma(z) \\
&= h_\gamma(z) \left(a_1^\gamma + a_3^\gamma |h_\gamma(z)|^2 + \cdots + a_{2n-1}^\gamma |h_\gamma(z)|^{2n-2} + \mathcal{O}(|h_\gamma(z)|^{2n}) \right) \,. \quad (2.15)
\end{aligned}
$$

The above coefficients only depend on the even moments of the measure ν_γ and on $\beta(\gamma)$, which is however a bounded, fixed function of γ. It is clear from the above considerations that our framework grants many degrees of freedom to the a priori measure, and indeed one expects a SPDE of the form 2.3 to describe the behavior of a large number of discrete models. It is not immediate however the fact that a solution exists, because the mononomials in (2.15) are not renormalized. The only way to renormalize them is to encode the renormalizing factors in the coefficients a_j (and hence in ν_γ), but it is not clear if this is possible because

(2.15) is not a general polynomial of degree $2n-2$ in m variable. We will discuss about the existence of a measure ν_γ, and its properties in Subsection 2.2.4.

Remark 2.2.6 In the calculation above, the first coefficient a_1^γ is given by

$$a_1^\gamma = \frac{\beta(\gamma)}{|h_\gamma(z)|^2}\int_S |\left\langle \beta h_\gamma(z),\eta\right\rangle|^2\nu_\gamma(d\eta)-1 = \frac{\beta(\gamma)}{m}\int_S |\eta|^2\nu_\gamma(d\eta)-1=\beta(\gamma)-1\ .$$

This shows the motivation behind the assumptions on the reference measure ν_γ and the choice for $\int_S |\eta|^2\nu_\gamma(d\eta)=m$. This exact calculation, together with (2.13), provides the form of the critical line in [SW16, Fig. 1].

We are now going to introduce the *scaling parameters* used to rescale space, time and height of the fluctuation.
For macroscopic coordinates $(x,t)\in\Lambda_\varepsilon\times[0,\infty)$, define the rescaled field

$$X_\gamma(x,t)\stackrel{\text{def}}{=}\delta^{-1}h_\gamma(x/\epsilon,t/\alpha) \tag{2.16}$$

where and α and δ are the scaling parameter of the time and the height of the field respectively. The relations between (ϵ,α,δ) have to be chosen in a specific way that we will describe below.
In macroscopic coordinates the effect of the Glauber dynamic on X_γ is given for $x\in\Lambda_\varepsilon$, $t\in[0,T]$ by the multidimensional SPDE

$$\begin{aligned}
dX_\gamma(x,t) &= \frac{\epsilon^2}{\gamma^2\alpha}\Delta_\gamma X_\gamma(x,t)+\alpha^{-1}\delta^{-1}K_\gamma *_\epsilon \Big(\Phi(\delta X_\gamma)-\delta X_\gamma\Big)(x,t^-)dt+dM_\gamma(x,t)\\
&=\frac{\epsilon^2}{\gamma^2\alpha}\Delta_\gamma X_\gamma(x,t)dt\\
&\quad+K_\gamma *_\epsilon X_\gamma\left(\frac{1}{\alpha}a_1^\gamma+\frac{\delta^2}{\alpha}a_3^\gamma|X_\gamma|^2+\cdots+\frac{\delta^{2n-2}}{\alpha}a_{2n-1}^\gamma|X_\gamma|^{2n-2}\right)(x,t^-)dt\\
&\quad+K_\gamma *_\epsilon E_\gamma(x,t)dt+dM_\gamma(x,t)\ .
\end{aligned} \tag{2.17}$$

Where $K_\gamma(x)\stackrel{\text{def}}{=}\epsilon^{-2}\kappa_\gamma(\epsilon x)$ is approximating a Dirac distribution, the convolution is defined $F*_\epsilon G(x)\stackrel{\text{def}}{=}\sum_{y\in\Lambda_\varepsilon}\epsilon^2F(x-y)G(y)$ and $\Delta_\gamma X_\gamma=\frac{\gamma^2}{\epsilon^2}\left(K_\gamma *_\epsilon X_\gamma-X_\gamma\right)$ is approximating a continuous Laplacian.
The form of the error term E_γ can be deduced from (2.15), and the value of the coefficients of the polynomial as well.

From (2.12) the martingale $M_\gamma(x,t) = \delta^{-1} m_\gamma(\epsilon^{-1}x, \alpha^{-1}t)$ has predictable quadratic variation given by the matrix

$$\left\langle {M_\gamma}^{(i)}(x,\cdot), {M_\gamma}^{(j)}(y,\cdot)\right\rangle_t = \frac{\epsilon^2}{\delta^2\alpha}\int_0^t \sum_{z\in\Lambda_\varepsilon} \epsilon^2 K_\gamma(x-z)K_\gamma(y-z)Q^{i,j}(s,z)ds \quad (2.18)$$

where the superscript (i) indicates the i-th component of a vector in $\mathbb{R}^m$ and

$$Q^{i,j}(s,z) \stackrel{\text{def}}{=} \int_S (\eta^{(i)} - \sigma^{(i)}_{\alpha^{-1}s}(\epsilon^{-1}z))(\eta^{(j)} - \sigma^{(j)}_{\alpha^{-1}s}(\epsilon^{-1}z))p^{h_\gamma(\epsilon^{-1}z,\alpha^{-1}s)}(d\eta) \quad (2.19)$$

where $p^{h_\gamma(\cdot,\cdot)}$ is defined in (2.8).

The conditions we have to impose to have in the limit, at least heuristically, the noise, the Laplacian and the $2n-1$ power of the field is to have

$$\frac{\epsilon^2}{\delta^2\alpha} \sim \frac{\epsilon^2}{\gamma^2\alpha} \sim \frac{\delta^{2n-2}}{\alpha} \sim 1\ ,$$

which yields the scaling

$$\epsilon = \gamma^n\ , \qquad \alpha = \gamma^{2n-2}\ , \qquad \delta = \gamma\ , \quad (2.20)$$

where $\epsilon^{-1} = N$ is an integer. From now on we will assume (2.20) to be satisfied.

It readily follows that the coefficients $a_1, a_3, \ldots, a_{2n-3}$ have to vanish in γ with a certain order. On top of that, the powers of the field X_γ need to be substituted by their Wick powers. From (2.20) and (2.17), define the coefficients $\tilde{\mathfrak{a}}^\gamma_1, \ldots, \tilde{\mathfrak{a}}^\gamma_{2n-1}$ and the polynomial $\tilde{\mathfrak{p}}_\gamma$ as

$$a_1 = \delta^{2n-2}\tilde{\mathfrak{a}}^\gamma_1\ , \qquad a_3 = \delta^{2n-4}\tilde{\mathfrak{a}}^\gamma_3 \qquad \cdots \qquad a_{2n-1} = \tilde{\mathfrak{a}}^\gamma_{2n-1}\ . \quad (2.21)$$

$$\tilde{\mathfrak{p}}_\gamma(X_\gamma(z,t)) = X_\gamma\left(\tilde{\mathfrak{a}}^\gamma_1 + \tilde{\mathfrak{a}}^\gamma_3|X_\gamma|^2 + \cdots + \tilde{\mathfrak{a}}^\gamma_{2n-1}|X_\gamma|^{2n-2}\right)(z,t) \quad (2.22)$$

where we recall that X_γ is a vector-valued field $X_\gamma : \Lambda_\varepsilon \mapsto \mathbb{R}^m$. In particular $\tilde{\mathfrak{p}}_\gamma(X_\gamma(\cdot,t)) : \Lambda_\varepsilon \mapsto \mathbb{R}^m$ and we shall refer to $\tilde{\mathfrak{p}}^{(j)}_\gamma(X_\gamma)$ when we are considering its j-th component.

With this scaling the error in (2.17) can be bounded by

$$|E_\gamma(x,t)| \le C\gamma^2|X_\gamma(x,t)|^{2n+1}\int_S e^{\gamma\beta|X_\gamma(x,t)||\eta|}\nu_\gamma(d\eta)\ . \quad (2.23)$$

Recall that in (2.17) all powers in the brackets are even powers and they can be rewritten as symmetric functions of $X^{(1)}_\gamma, \ldots, X^{(m)}_\gamma$ the components of the vector X_γ. It is then clear that the renormalized polynomial would be a symmetric function of $X^{(1)2}_\gamma, \ldots, X^{(m)2}_\gamma$, and we will prove that it is possible to renormalize (2.22) just making use of the dependency on

γ of the coefficients $\tilde{\mathfrak{a}}_i^\gamma$ for $i = 1, \dots, 2n-1$. We will describe how to do so in a systematic way in Remark 2.2.7, defining $\mathfrak{a}_{2k-1}^\gamma$ in (2.35), and we will describe it more carefully in Section 2.5, where we will perform such renormalization in detail.
Denote the polynomial arising after the renormalization with

$$\mathfrak{p}_\gamma^{(j)}(X_\gamma(z,t)) = \mathfrak{a}_1^\gamma X_\gamma^{(j)}(z,t) + \mathfrak{a}_3^\gamma : X_\gamma^{(j)}|X_\gamma|^2 : (z,t) + \cdots + \mathfrak{a}_{2n-1}^\gamma : X_\gamma^{(j)}|X_\gamma|^{2n-2} : (z,t) \tag{2.24}$$

and $\mathfrak{p}$ the polynomial in (2.3) that is expected to be found in the limit

$$\mathfrak{p}^{(j)}(X(z,t)) = \mathfrak{a}_1 X^{(j)}(z,t) + \mathfrak{a}_3 : X^{(j)}|X|^2 : (z,t) + \cdots + \mathfrak{a}_{2n-1} : X^{(j)}|X|^{2n-2} : (z,t) \ . \tag{2.25}$$

In the above equations $: \ X_\gamma^{(j)}|X_\gamma|^{2j} \ :$ and $: \ X^{(j)}|X|^{2j} \ :$ have to be understood as the renormalized counterparts of $X_\gamma^{(j)}|X_\gamma|^{2j}$ and $X^{(j)}|X|^{2j}$ respectively.
With the above notations (2.17) is rewritten as

$$dX_\gamma(x,t) = (\Delta_\gamma X_\gamma(x,t) + K_\gamma *_\epsilon \tilde{\mathfrak{p}}_\gamma(X_\gamma(\cdot,t))(x) + K_\gamma *_\epsilon E_\gamma(x,t))\, dt + dM_\gamma(x,t)$$

which looks already like the equation satisfied by the limiting process (2.3). Define now the semigroup $P_t^\gamma \stackrel{\text{def}}{=} e^{t\Delta_\gamma}$ characterizated by its action on functions $f : \Lambda_\varepsilon \to \mathbb{R}$

$$P_t^\gamma f(x) = \frac{1}{4} \sum_{\omega \in \Lambda_N} e^{-t\hat{\Delta}_\gamma(\omega)} \hat{f}(\omega) e_\omega(x) \tag{2.26}$$

Therefore we expect the solution to have the form

$$X_\gamma(\cdot,t) = P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma *_\epsilon \tilde{\mathfrak{p}}_\gamma(X_\gamma)(\cdot,s)\, ds + \int_0^t P_{t-s}^\gamma dM_\gamma(\cdot,s) + \dots \ . \tag{2.27}$$

The quadratic variation of the martingale is going to play a major role in the construction of the Wick powers of the process. In Section 2.3 we are going to show that

$$\mathbb{E}\left\langle \int_0^t P_{t-s}^\gamma dM_\gamma^{(j)}(x,s), \int_0^t P_{t-s}^\gamma dM_\gamma^{(j)}(x,s) \right\rangle$$

is well approximated, in the sense of Proposition 2.4.4, by

$$\begin{aligned} 2\int_0^t \sum_{z\in\Lambda_\varepsilon} \epsilon^2 \left|P_{t-s}^\gamma K_\gamma(x-z)\right|^2 ds \\ &= \frac{t}{2} + \frac{1}{4} \sum_{\omega\in\Lambda_N\setminus\{0\}} \frac{|\hat{K}_\gamma(\omega)|^2}{\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))} \left(1 - e^{-2t\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))}\right) \ . \end{aligned}$$

For fixed t, the above quantity is diverging as $\gamma \to 0$ essentially like

$$\mathfrak{c}_\gamma \stackrel{\text{def}}{=} \sum_{\substack{\omega\in\mathbb{Z}^2 \\ 0<|\omega|_\infty\leq\epsilon^{-1}}} \frac{|\hat{K}_\gamma(\omega)|^2}{4\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))} \tag{2.28}$$

and it is easy to see that $\mathfrak{c}_\gamma \sim \log(\gamma^{-1}) \nearrow \infty$. This fact is expected since the limiting process has solutions in a distributional space. As we shall see in Subsection 2.2.2, the factor (2.28) will be used to renormalize the mononomials in (2.22) since it is related to $\mathbb{E}[|X_\gamma^{(j)}|^2]$ for each $j = 1, \ldots, m$.
In order to go from (2.24) to (2.25) we need to introduce the Hermite polynomial.

2.2.2 Hermite Polynomials and renormalization

The aim of this subsection is to clarify the way we renormalize the polynomial arising from the discrete model and to recall some general facts for later reference.
Recall that for a multiindex $\mathbf{k} = (k_1, \cdots, k_m) \in \mathbb{N}^n$ and vector $\bar{x} = (x_1, \ldots, x_m)$ and positive definite matrix $\bar{T} = (T_{i,j})_{i,j=1}^m$ the multivariate Hermite polynomial $H_\mathbf{k}$ are defined as the coefficients of the Taylor expansion in λ

$$\exp\left\{\sum_{j=1}^m \lambda_j x_j - \frac{1}{2}\sum_{i,j=1}^m \lambda_i\lambda_j T_{i,j}\right\} = \sum_{k_1,\ldots,k_m\geq 0} \frac{H_\mathbf{k}(\bar{x},\bar{T})}{\mathbf{k}!}\lambda^\mathbf{k} \tag{2.29}$$

where we used the convention $\lambda^\mathbf{k} = \lambda^{k_1}\cdots\lambda^{k_m}$ and $\mathbf{k}! = k_1!\cdots k_m!$.
From the above definition it follows that the Hermite polynomials satisfy the following proprieties

$$\begin{aligned}
&\partial_{x_j} H_{(k_1,\ldots,k_m)}(\bar{x},\bar{T}) = k_j H_{(k_1,\ldots,k_j-1\ldots,k_m)}(\bar{x},\bar{T}) \\
&\partial_{T_{i,j}} H_{(k_1,\ldots,k_m)}(\bar{x},\bar{T}) = -\frac{1}{2}\begin{cases} k_i k_j H_{(k_1,\ldots,k_i-1\ldots,k_j-1\ldots,k_m)}(\bar{x},\bar{T}) & \text{for } i\neq j \\ (k_j-1)k_j H_{(k_1,\ldots,k_j-2\ldots,k_m)}(\bar{x},\bar{T}) & \text{for } i=j \end{cases}
\end{aligned}$$

and that for $\bar{v}\in\mathbb{R}^m$

$$H_\mathbf{k}(\bar{x}+\bar{v},\bar{T}) = \sum_{\substack{\mathbf{a}\in\mathbb{N}^m \\ \mathbf{a}\leq\mathbf{k}}} \binom{\mathbf{k}}{\mathbf{a}} \bar{v}^\mathbf{a} H_{\mathbf{k}-\mathbf{a}}(\bar{x},\bar{T})\ . \tag{2.30}$$

Assume now that $\bar{T} = \mathfrak{c}I_m$ where I_m is the identity matrix in $\mathbb{R}^{m\times m}$ and $\mathfrak{c} > 0$ we abuse the notation writing instead of $H_{\mathbf{k}}(\bar{x}, \mathfrak{c}I_m)$

$$H(x_1^{k_1} \cdots x_m^{k_m}, \mathfrak{c}I_m) = \prod_{j=1}^{m} H_{k_j}(x_j, \mathfrak{c}) \tag{2.31}$$

where H_{k_j} is the unidimensional Hermite polynomial.
Let $G_{\mathfrak{c}}(\bar{x}) = (2\pi\mathfrak{c})^{-\frac{m}{2}} \exp\{-\frac{\|\bar{x}\|^2}{2\mathfrak{c}}\}$ be the density of the multivariate Gaussian with mean zero and covariance matrix $\mathfrak{c}I$. The Hermite polynomials (2.31) from a complete orthogonal basis of $L^2(G_{\mathfrak{c}}(\bar{x})d\bar{x})$ where the orthogonality is a consequence of (2.29)

$$\int_{\mathbb{R}^m} H(\bar{x}^{\mathbf{k}}, \mathfrak{c}I_m)H(\bar{x}^{\mathbf{h}}, \mathfrak{c}I_m))G_{\mathfrak{c}}(\bar{x})d\bar{x} = \mathbf{k}!\mathfrak{c}^{|\mathbf{k}|}\delta_{\mathbf{h}=\mathbf{k}} .$$

We extend by linearity the above expression to any polynomial in m variables setting

$$H\left(\sum_{\mathbf{a}\in\mathbb{N}^m} b_{\mathbf{a}} x_1^{a_1} \cdots x_m^{a_m}, \mathfrak{c}I_m\right) = \sum_{\mathbf{a}\in\mathbb{N}^m} b_{\mathbf{a}} H(x_1^{a_1} \cdots x_m^{a_m}, \mathfrak{c}I_m) .$$

The notation above can be justified with the following expression for the n-th Hermite polynomial

$$H(\bar{x}^{\mathbf{k}}, \mathfrak{c}I_m) = e^{-\frac{\mathfrak{c}}{2}\Delta}\bar{x}^{\mathbf{k}} = \left(1 - \frac{\mathfrak{c}}{2}\Delta + \frac{\mathfrak{c}^2}{8}\Delta^2 - \dots\right)\bar{x}^{\mathbf{k}} . \tag{2.32}$$

Remark 2.2.7 We will use in the following sections the fact that the renormalization for the polynomial $x^{(i)}|\bar{x}|^{2n}$ is given by a similar polynomial

$$H(x^{(i)}|\bar{x}|^{2n}, \mathfrak{c}I_m) = e^{-\frac{\mathfrak{c}}{2}\Delta}x^{(i)}|\bar{x}|^{2n} = \sum_{k=1}^{n} b_k(\mathfrak{c})x^{(i)}|\bar{x}|^{2k} ,$$

this can be seen using the fact that

$$
\begin{aligned}
\Delta x^{(i)}|\bar{x}|^{2n} &= \left(\sum_{j=1}^{m}\partial_j^2\right) x^{(i)}|\bar{x}|^{2n} \\
&= \partial_i|\bar{x}|^{2n} + 2n\partial_i x^{(i)2}|\bar{x}|^{2n-2} + \sum_{j\neq i} 2n\partial_j x^{(i)}x^{(j)}|\bar{x}|^{2n-2} \\
&= 2nx^{(i)}|\bar{x}|^{2n-2} + 4nx^{(i)}|\bar{x}|^{2n-2} + 2n(2n-2)x^{(i)3}|\bar{x}|^{2n-4} \\
&+ \sum_{j\neq i} 2nx^{(i)}|\bar{x}|^{2n-2} + 2n(2n-2)x^{(i)}x^{(j)2}|\bar{x}|^{2n-4} \\
&= 2n(m+2n)x^{(i)}|\bar{x}|^{2n-2} \ .
\end{aligned}
\tag{2.33}
$$

The issue that we are going to discuss now concerns the renormalization procedure to highlight the Wick powers of the field X_γ.

$$
\begin{aligned}
\tilde{\mathfrak{p}}_\gamma^{(j)}(X) &= \sum_{k=0}^{n-1}\tilde{\mathfrak{a}}_{2j+1}^\gamma e^{\frac{\mathfrak{c}}{2}\Delta_X}e^{-\frac{\mathfrak{c}}{2}\Delta_X}X^{(j)}|X|^{2k} = \sum_{k=0}^{n-1}\tilde{\mathfrak{a}}_{2j+1}^\gamma e^{\frac{\mathfrak{c}}{2}\Delta_X}H(X^{(j)}|X|^{2k},\mathfrak{c}) \\
&= \sum_{k=0}^{n-1}\left(e^{\frac{\mathfrak{c}}{2}\Delta_X^*}\tilde{\mathfrak{a}}^\gamma\right)_{2j+1}H(X^{(j)}|X|^{2k},\mathfrak{c}) \ .
\end{aligned}
\tag{2.34}
$$

Hence we define the coefficients in (2.24) as

$$
\mathfrak{a}_{2k+1}^\gamma \stackrel{\text{def}}{=} \left(e^{\frac{\mathfrak{c}_\gamma}{2}\Delta_X^*}\tilde{\mathfrak{a}}^\gamma\right)_{2k+1} \ . \tag{2.35}
$$

It is immediate to see from its definition that the exponential of Δ_X^* is a well defined operation on the space l_0^∞ of the sequences which are eventually zero.
The above calculation shows how is possible to write the polynomial $\mathfrak{p}_\gamma^{(j)}(X)$ in (2.22) as

$$
\tilde{\mathfrak{p}}_\gamma^{(j)}(X) \ = \ \mathfrak{a}_1^\gamma X^{(j)} + \mathfrak{a}_3^\gamma H(X^{(j)}|X|^2,\mathfrak{c}_\gamma) + \cdots + \mathfrak{a}_{2n-1}^\gamma H(X^{(j)}|X|^{2m-2},\mathfrak{c}_\gamma) \ . \tag{2.36}
$$

Remark 2.2.8 As a consequence of this definition and the assumption above, we can provide a formula for the value of $\beta(\gamma)$ and its discrepancy from its critical value 1. From Remark 2.2.6, the expression for a_1^γ in (2.21), (2.20) and (2.35) we have

$$
\beta(\gamma) = 1 + a_1^\gamma = 1 + \alpha\tilde{\mathfrak{a}}_1^\gamma = 1 + \alpha\left(e^{-\frac{\mathfrak{c}_\gamma}{2}\Delta_X^*}\mathfrak{a}^\gamma\right)_1
$$

by the fact that $\mathfrak{c}_\gamma$ is diverging as $\log(\gamma^{-1})$ and assumption (M1) we have

$$
\beta(\gamma) = 1 + \alpha\left(e^{-\frac{\mathfrak{c}_\gamma}{2}\Delta_X^*}\mathfrak{a}\right)_1 + \mathcal{O}(\alpha\gamma^{\lambda_0}\mathfrak{c}_\gamma^n) \tag{2.37}
$$

where $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{2n-1}, 0, \dots)$ are the coefficients of the limiting polynomial. It is immediate to see that this value coincide with the choice of the critical temperature in [MW17a] and in [SW16] (see Remark 2.2.5).

2.2.3 Assumptions over ν_γ and the initial condition

We are now able to formulate a complete list of assumptions for the reference measure ν_γ and on the initial distribution of the spins $\{\sigma_x(0)\}_{x\in\Lambda_N}$ in the same section for convenience.

Assumption 2.2.9 *The measure ν_γ on $S \subseteq \mathbb{R}^m$ is isotropic, has exponential moments of any order, i.e. for any $\theta > 0$*

$$\int_S e^{\theta|\eta|}\nu_\gamma(d\eta) < \infty \tag{M0}$$

uniformly in γ in a neighborhood of the origin, and moreover $\int_S |\eta|^2\nu_\gamma(d\eta) = m$.

The value of m for the variance of the measure is essentially arbitrarily, a different choice would effect the coefficient in front of the noise in (2.3) and hence the definition of the factor $\mathfrak{c}_\gamma$. With this choice, we kept the definition of $\mathfrak{c}_\gamma$ as in [MW17a].

Recall the coefficient used in (2.24), defined in (2.35). The following is a condition, given in a very implicit form, on the form of the moment generating function of the measure ν_γ.

Assumption 2.2.10 *Recall the scaling* (2.20)*, the formal Taylor expansion of* $\Phi(\lambda)$

$$\Phi(\lambda) = \lambda\left(1 + a_1^\gamma + a_3^\gamma|\lambda|^2 + \dots + a_{2n-1}^\gamma|\lambda|^{2n-2} + \sum_{j>n} a_{2j-1}^\gamma|\lambda|^{2j}\right)$$

and the definitions

$$\tilde{\mathfrak{a}}_{2j-1}^\gamma \overset{def}{=} \gamma^{-2n+2j}a_{2j-1}^\gamma\,, \qquad \mathfrak{a}_{2k+1}^\gamma \overset{def}{=} \left(e^{\frac{\mathfrak{c}_\gamma}{2}\Delta_X^*}\tilde{\mathfrak{a}}^\gamma\right)_{2k+1}\,.$$

There exists $c_0 > 0$ and $\lambda_0 > 0$ such that

$$\sup_{k=1,\dots,n}\left|\mathfrak{a}_{2k-1}^\gamma - \mathfrak{a}_{2k-1}\right| \le c_0\gamma^{\lambda_0}\,. \tag{M1}$$

The next assumption is necessary to define the limiting dynamic for any time.

Assumption 2.2.11 *The leading coefficient of the limiting polynomials $\mathfrak{p}^{(j)}$ (see* (M1)*) of degree $2n-1$ satisfy*

$$\mathfrak{a}_{2n-1} < 0\,. \tag{M2}$$

In order to prove the convergence of the linear and non linear dynamic, we now state the hypothesis on the initial distribution of the spins $\sigma(0)$. Two hypothesis are needed in order to prove the result: they are mainly needed to control the processes uniformly in γ. The first one concerns the regularity of the initial profile:

Assumption 2.2.12 *Let $X^0 \in \mathcal{C}^{-\nu}(\mathbb{T}^2)$ for any $\nu > 0$.*

$$\lim_{\gamma\to 0} \mathbb{E}\left\|\delta^{-1}h_\gamma(\cdot,0) - X^0\right\|_{\mathcal{C}^{-\nu}} = 0\ . \tag{I1}$$

This will be used to control the contribution of the initial condition to the $\mathcal{C}^{-\nu}$ norm of the process. The second assumption is used to get a uniform control over the quantity (2.19). It is a control over the starting measure.

Assumption 2.2.13 *For all $p \geq 1$ there exists a $\gamma_0 > 0$ such that*

$$\sup_{\gamma<\gamma_0} \mathbb{E}\left[\|\sigma(0)\|_{L^\infty}^p\right] < \infty\ . \tag{I2}$$

In the work [MW17a] the (I2) assumption is not needed since the state space of the spins is a compact set.
Condition (I2) can be relaxed to the assumption that the initial condition has all moments pointwise, i.e.

$$\sup_{\gamma<\gamma_0}\sup_{z\in\Lambda_\varepsilon} \mathbb{E}\left[|\sigma_z(0)|^p\right] < \infty\ . \tag{I2'}$$

With assumption (I2') and the monotonicity of L^p norms one can prove (I2) up to any small negative power of γ:

$$\mathbb{E}\left[\|\sigma(0)\|_{L^\infty}^p\right] \lesssim \gamma^{-\kappa}\ . \tag{2.38}$$

2.2.4 Existence of a reference measure ν_γ

We saw in Subsection 2.2.2 that the limiting polynomial comes from a combination of the moments (or cumulants) of the reference measure ν_γ. We will now start from a given renormalized polynomial and show the existence of an *a priori* measure producing such a polynomial. The problem of find a measure with a given sets of moments is known in the literature as the "moment problem" (see [Akh65]). For the equation to make sense in the limit, the polynomial has to be renormalized as in [DPD03] with a renormalization constant $\mathfrak{c}_\gamma$ diverging as $\gamma \to 0$. The precise value of $\mathfrak{c}_\gamma$ has been given in (2.28) and it is not important at this stage. The only fact that we will use is that the divergence is logarithmic as

$\gamma \to 0$, hence slower that any negative power of γ. Recall the expansion

$$\begin{aligned}&\frac{1}{\alpha\delta}\left(\Phi(h_\gamma(z,t)) - h_\gamma(z,t)\right)\\&= X_\gamma\left(\frac{1}{\alpha}a_1^\gamma + \frac{\delta^2}{\alpha}a_3^\gamma|X_\gamma|^2 + \cdots + \frac{\delta^{2n-2}}{\alpha}a_{2n-1}^\gamma|X_\gamma|^{2n-2}\right)(z,t) + \frac{\delta^{2n}}{\alpha}\mathcal{O}(|X_\gamma(z,t)|^{2n+1}).\end{aligned}$$

The scaling (2.20) entails the fact that the coefficients $a_1^\gamma, \dots, a_{2n-3}^\gamma$ are vanishing at a suitable rate. We know, however, from the form (2.14) that they are identically zero as soon as ν_γ shares the first $2n-2$ moments with a m dimensional Gaussian random variable with a suitable covariance matrix. In fact (2.14) tells that the coefficient depends on the difference between the cumulants of the measure ν_γ minus the cumulants of a multivariate Gaussian random variable.

Proposition 2.2.14 *Let C_γ any positive sequence of renormalization constants diverging logarithmically as $\gamma \to 0$.*
Let $b_\gamma \to b \in \mathbb{R}^+$ a sequence of positive real numbers.
Let now $n \geq 2$ and $\mathfrak{a}_1, \dots, \mathfrak{a}_{2n-1} \in \mathbb{R}$ with $\mathfrak{a}_j = 0$ if j is even.
For all $m \in \mathbb{N}$ and $\gamma < \gamma_0$ small enough for any $\mathfrak{a}_{2n-1} < 0$ small enough in absolute value, there exists a family of rotational invariant measures $\{\mu_\gamma\}_{\gamma<\gamma_0}$ over $\mathbb{R}^m$ such that

- *$\forall \gamma < \gamma_0$, μ_γ have all exponential moments*
- *The sequence μ_γ is weakly convergent as $\gamma \to 0$*
- *For $j = 1, \dots, m$ the polynomial obtained in* (2.36)*, using the measure μ_γ and the inverse temperature b_γ is*

$$\mathfrak{a}_1 X^{(j)} + \mathfrak{a}_3 H(X^{(j)}|X|^2, C_\gamma) + \cdots + \mathfrak{a}_{2n-1}H(X^{(j)}|X|^{2m-2}, C_\gamma)\ .$$

Moreover it is possible to choose the family μ_γ to be supported in the same compact set of $\mathbb{R}^m$.

Proof. It is clear that it suffices to prove the theorem above in the case $b_\gamma = 1$ for all γ. We first describe conditions over the moments (or the cumulants) of the measure. From (2.21), (2.35) and (2.14) we have that μ_γ has to satisfy, for $\lambda \in \mathbb{R}^m$

$$\log\left(\int_S e^{\langle\lambda,\eta\rangle}\mu_\gamma(d\eta)\right) = \frac{1}{2}|\lambda|^2 + \sum_{j=1}^{n}\frac{\alpha}{\delta^{2j-2}}\left(e^{-\frac{C_\gamma}{2}\Delta_X^*}\mathfrak{a}\right)_{2j-1}\frac{1}{2j}|\lambda|^{2j} + \mathcal{O}(|\lambda|^{2n+2})$$

where $e^{-\frac{C_\gamma}{2}\Delta^*_X}$ is the inverse of the operator described in Subsection 2.2.2 and the sequence $\mathfrak{a}$ is extended to be zero after the $2n-1$-th place. For γ small the above polynomial is a perturbation of

$$\frac{1}{2}|\lambda|^2 + \frac{\mathfrak{a}_{2n-1}}{2n}|\lambda|^{2n} + \mathcal{O}(|\lambda|^{2n+2})$$

taking the exponential we see that the μ_γ can be seen as a small perturbation of a multivariate Gaussian random variable, up to the $2n-2$-th moment (recall that $\alpha = \delta^{2n-1}$ by (2.20)). Here we are using the fact that C_γ has a logarithmic divergence in γ. For λ with modulus 1, the $2n$-th moment is given by

$$\int_S |\langle \lambda, \eta \rangle|^{2n} \mu_\gamma(d\eta) = \frac{2n!}{n!2^n} + (2n-1)!\mathfrak{a}_{2n-1}$$

since we aim to produce an isotropic measure, it is sufficient then to prove the existence of a univariate distribution having the first $2n$ even moments equal to

$$m^\gamma_{2j} = \begin{cases} \frac{2j!}{j!2^j} + o(1) & \text{if } j < n \\ \frac{2n!}{n!2^n} + (2n-1)!\mathfrak{a}_{2n-1} + o(1) & \text{if } j = n \end{cases} \tag{2.39}$$

for $\gamma \to 0$. Since we have the freedom to chose the higher moments, we shall do that later. Such a measure is known to exists (see [Akh65]) if and only if the moment matrix is positive definite i.e. for all choice complex numbers $\{z_0, z_1, \ldots, z_p\}$

$$\sum_{i_1,i_2=0}^{p} m^\gamma_{i_1+i_2} \bar{z}_{i_1} z_{i_2} \geq 0 \,. \tag{2.40}$$

It is easy to see the necessity of such condition, since (2.40) is the expectation of the square of a polynomial in the random variable. Condition (2.40) is satisfied for m^γ_j if it is satisfied with a strict inequality for $m_j \stackrel{\text{def}}{=} \lim_{\gamma\to 0} m^\gamma_j$. It is easy to see that for a standard Normal $U \sim \mathcal{N}(0,1)$ (2.40) holds for a strict inequality if $\sum_{i=0}^p |z_i|^2 > 0$

$$\mathbb{E}\left[\left|z_0 + z_1 U + z_2 U^2 + \cdots + z_p U^p\right|^2\right] > 0 \,.$$

It is immediate to conclude that there exists a negative value of $\mathfrak{a}_{2n-1}$ and γ small enough such that the above inequality is satisfied for the collection of moments given by (2.39). The values of $\mathfrak{a}_{2n-1}$ that guarantee condition (2.40) are given by the inequality

$$\mathfrak{a}_{2n-1} > -\frac{D_{2n}}{(2n-1)!D_{2n-1}} \,. \tag{2.41}$$

If we denote with $D_p \stackrel{\text{def}}{=} \det\left(m^G_{i_1+i_2}\right)^p_{i_1,i_2=0} > 0$ the determinant of the moment matrix of

the Gaussian random variable U.
In order to complete the proof it is sufficient to complete the list of the moments with arbitrary values satisfying 2.40. It is always possible to find such a sequence of moments because each moment is asked to satisfy an inequality similar to 2.41 which admits trivially a solution.
This implies the existence of a measure with the prescribed moments.
We might chose, in particular, at some $p > n$ to have the left hand side of (2.40) equal to 0 for all γ. This implies that the random variable annihilates a certain nonnegative polynomial and therefore it is supported on the set of the real zeros of such polynomial, which is a finite set, hence a bounded set. This proves the last claim. □

2.2.5 Stopping time for the dynamic

As announced in Remark 2.2.3, we are now ready to define a stopping time $\tau_{\gamma,\mathfrak{m}}$ for the macroscopic dynamic defined above. Fix a positive $\nu > 0$ and a value $\mathfrak{m} > 1$, and let

$$\tau_{\gamma,\mathfrak{m}} = \inf \left\{ t \geq 0 | \, \|X_\gamma(t,\cdot)\|_{\mathcal{C}^{-\nu}} \geq \mathfrak{m} \right\} \, . \tag{2.42}$$

Following [MW17a] we will not work directly with the process introduced in Subsection 2.2.1, but with a process whose jump distribution is given by (for macroscopic coordinates $(x,s) \in \Lambda_\varepsilon \times [0,T]$)

$$p_{\mathfrak{m}}(x, s^-, \sigma)(d\eta) = \begin{cases} p^{h_\gamma(\epsilon^{-1}x, \alpha^{-1}s^-)}(d\eta) & \text{if } s \leq \alpha^{-1}\tau_{\gamma,\mathfrak{m}} \\ \nu_\gamma(d\eta) & \text{if } s > \alpha^{-1}\tau_{\gamma,\mathfrak{m}} \end{cases} \tag{2.43}$$

In particular, $p_{\mathfrak{m}}(x, s^-, \sigma)(d\eta_x)$ doesn't depend on the current configuration when $s > \alpha^{-1}\tau_{\gamma,\mathfrak{m}}$ and for general $\mathfrak{m} \geq 0$, $k > 0$ and $s > 0$, from assumption (M0)

$$\int_S |\eta|^k p_{\mathfrak{m}}(z, s^-, \sigma)(d\eta) \leq C(k, \mathfrak{m}) \, . \tag{2.44}$$

The process with jump distribution (2.43) coincides with the process defined in Section 2.2 up to the stopping time, after which it still follows a Glauber dynamic, but with respect to the Gibbs measure at infinite temperature ($\beta = 0$).

Remark 2.2.15 The reason behind the introduction of the stopping time is to guarantee a control over the norm of the fluctuation field because the quadratic variation of the linear part Z_γ depends on the whole fluctuation field X_γ. It can be proven, following the same proof as [MW17a, Theorem 2.1], that for all arbitrarily small $\zeta > 0$, there exists $\mathfrak{m} > 1$ such that

$$\limsup_{\gamma\to 0} \mathbb{P}[\tau_{\gamma,\mathfrak{m}} \le T] \le \zeta$$

that allows us to use "a posteriori" the dynamic defined in (2.43).
A consequence of this fact is that in order to prove the convergence in distribution for X_γ it is sufficient to show that, $\forall \mathfrak{m} > 1$, and all continuous bounded $F : \mathcal{D}([0,T], \mathcal{C}^{-\nu}) \to \mathbb{R}$

$$\limsup_{\gamma\to 0} \mathbb{E}\left[|F(X_\gamma) - F(X)|\mathbf{1}_{\{\tau_{\gamma,\mathfrak{m}} > T\}}\right] = 0 \ .$$

This means that we can always assume to work with the stopped dynamic, and we will do so.

For the new process, define $Q_\mathfrak{m}$ as in (2.19)

$$\begin{aligned} Q^{i,j}_\mathfrak{m}(s^-, z) = &\int_S (\eta^{(i)} - \sigma^{(i)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-))(\eta^{(j)} - \sigma^{(j)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-))p_\mathfrak{m}(z, s^-, \sigma)(d\eta) \\ &= \delta_{i,j} + \sigma^{(i)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-)\sigma^{(j)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-) + err_\gamma(\mathfrak{m}, \sigma, z, s^-) \end{aligned} \tag{2.45}$$

with

$$|err_\gamma(\mathfrak{m}, \sigma, z, s^-)| \lesssim \gamma^{1-\nu} \begin{cases} |\sigma^{(i)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-)| + |\sigma^{(j)}_{\lfloor \epsilon^{-1} z\rfloor}(\alpha^{-1}s^-)| & \text{if } s < \alpha^{-1}\tau_{\gamma,\mathfrak{m}} \\ 0 & \text{otherwise} \end{cases}$$

where the proportionality constant might depend on $\mathfrak{m}$.

Remark 2.2.16 In order to keep the notation cleaner, since we will always use the stopped dynamic for the rest of the paper we will abuse the notation omitting from σ, h_γ and all the other fields the dependence on $\mathfrak{m}$.

We have the following proposition.

Proposition 2.2.17 *Let err_γ be the error term in* (2.45)*. For all $\lambda > 0$ and $q > 1$ there exists $C = C(q, \mathfrak{m}, \lambda, T)$, depending on the constant in* (I2')*, such that, for some $\gamma_0 > 0$*

$$\sup_{0<\gamma\le\gamma_0} \sup_{0\le s\le T} \sup_{z\in\Lambda_\varepsilon} s^\lambda \mathbb{E}\left[\left|err_\gamma(\mathfrak{m}, \sigma, z, s^-)\right|^q\right] \le C\left(\gamma^{q(1-\nu)} + \alpha^\lambda\right)$$

where s, T are macroscopic times. Moreover, there exists $C = C(\mathfrak{m}, q, T)$ such that

$$\sup_{0<\gamma\le\gamma_0} \sup_{0\le s\le T} \sup_{z\in\Lambda_\varepsilon} \mathbb{E}\left[|Q^{i,j}_\mathfrak{m}(s,z)|^q\right] \le C \ .$$

Proof. It is sufficient to notice that the Radon-Nikodym derivative

$$e^{-2\gamma^{1-\nu}\mathfrak{m}|\eta|} \leq dp_{\mathfrak{m}}(z,s^-,\sigma)/d\nu_\gamma(d\eta) \leq e^{2\gamma^{1-\nu}\mathfrak{m}|\eta|}$$

and that the measure ν_γ has exponential moments by (M0). Then

$$\begin{aligned} &r^\lambda \mathbb{E}\left[\left|\sigma^{(i)}_{\epsilon^{-1}y}(\alpha^{-1}r)\right|^q\right] \\ &\qquad \leq Cr^\lambda \mathbb{P}(T_0 \geq r) + r^\lambda \mathbb{P}(T_0 < r)\int_S |\eta^{(i)}|^q p_{\mathfrak{m}}(y,r^-,\sigma)(d\eta) \leq Cr^\lambda e^{-\alpha^{-1}r} + C \end{aligned}$$

for $0 \leq r \leq T$, $y \in \Lambda_N$ and where T_0 it the first (macroscopic) time that the spin in y jumps. To go from the first line to the second one we used the assumption (I2') on the initial condition. □

2.2.6 Limiting SPDE

In this section we define the solution to the limiting equation (2.3), a multidimensional version of the Φ^{2n}_2 equation, which will be the limit for the discrete process introduced in Section 2.2.

For $j = 1,\dots,m$, each $X^{(j)}$ turns out to be a process with values in the Besov space of negative regularity $\mathcal{C}^{-\nu}(\mathbb{T}^2)$ for any $\nu > 0$. Recall that $\mathcal{C}^\alpha$ is defined as the closure of the space of smooth functions under the norm

$$\|g\|_{\mathcal{C}^\alpha} \stackrel{\text{def}}{=} \sup_{k\geq -1} 2^{\alpha k}\,\|\delta_k g\|_{L^\infty(\mathbb{T}^2)}$$

where δ_k is the k-th Paley-Littlewood projection (see Section 1.2.2 of Chapter 1 for the details of the construction in our case). The multivalued stochastic quantization equation in two dimension is given by

$$dX^{(j)}(\cdot,t) = \Delta X^{(j)}(\cdot,t)dt + \mathfrak{p}^{(j)}(X)(\cdot,t)dt + \sqrt{2}dW^{(j)}(t) \tag{2.46}$$

with initial conditions $X^0 \in \mathcal{C}^{-\nu}(\mathbb{T}^2;\mathbb{R}^m)$. The processes $W^{(j)}$ are m independent white noises on $\mathbb{T}^2$ and $\mathfrak{p}^{(j)}$ are odd renormalized polynomials of degree $2n-1$ of the form 2.25 satisfying assumption (M2).

The existence and uniqueness theory behind (2.46) follows from the work of [DPD03] and [TW16]. The analysis of equation (2.46) has already been performed in our context by [MW17a] for an odd polynomial of degree 3 and $m = 1$ and in [SW16] in case of an odd polynomial of any degree and $m = 1$. The extension to the multidimensional case is straightforward, but we will present it in order to fix some notations and definitions useful in

Section 2.5.

Let $W_\epsilon^{(j)}$ be a smooth approximation of the white noise, given by truncating the Fourier modes with frequencies $|\omega|_\infty > \epsilon^{-1}$. Assume $Z_\epsilon^{(j)}$ for $j = 1, \dots, m$ to be the smooth solution of the heat equation on the torus $\mathbb{T}^2$ with Gaussian noise $W_\epsilon^{(j)}$

$$\begin{cases} \partial_t Z_\epsilon^{(j)}(t) & = \Delta Z_\epsilon^{(j)}(t) + \sqrt{2} dW_\epsilon^{(j)}(t) \\ Z_\epsilon^{(j)}(t) & = 0 \end{cases} \tag{2.47}$$

For a positive integer $k \in \mathbb{N}$, define the Wick power $Z_\epsilon^{(j):k:}(t,x) \stackrel{\text{def}}{=} H_k(Z_\epsilon^{(j)}(t,x), \mathfrak{c}_\epsilon(t))$ where

$$\mathfrak{c}_\epsilon(t) = \mathbb{E}[(Z(t,0)^{(j)})^2] = \frac{t}{2} + \sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{1}{4\pi^2 |\omega|^2} \left(1 - e^{-2t\pi^2|\omega|^2}\right)$$

and extend the above definition to a general multiindex $\mathbf{k} \in \mathbb{N}^m$ using the definition of multidimensional Hermite polynomial in the case of independent components (2.31)

$$Z_\epsilon^{:\mathbf{k}:}(t,x) \stackrel{\text{def}}{=} H_{\mathbf{k}}\left((Z_\epsilon^{(1)}(t,x), \dots, Z_\epsilon^{(m)}(t,x)), \mathfrak{c}_\epsilon(t) I_m\right) = \prod_{j=1}^m H_{k_j}(Z_\epsilon^{(j)}(t,x), \mathfrak{c}_\epsilon(t)) \ .$$

Remark 2.2.18 The Wick powers $Z_\epsilon^{:\mathbf{k}:}$ do have Fourier modes of frequencies of order $|\omega|_\infty \sim |\mathbf{k}|\epsilon^{-1}$. This will be important when dealing with the nonlinear process. This is exactly the reason behind the definition of lower and higher truncation of the process (2.54).

We will now state a result which is a multidimensional dynamical version of [DPD03, Lemma 3.2] and [MW17a, Prop 3.1]. The proof of the result follows essentially from [MW17a], with the use of the independence of the components.

Proposition 2.2.19 *For $T > 0$, $\nu > 0$ and $\mathbf{k} \in \mathbb{N}^m$, the stochastic processes $Z_\epsilon^{:\mathbf{k}:}$ converges a.s. and in any stochastic L^p space in the metric of $\mathcal{C}([0,T], \mathcal{C}^{-\nu})$.*
We will refer to this limit with $Z^{:\mathbf{k}:}(t,\cdot)$.

The solution of the linear equation (2.47) in $\mathbb{T}^2$ started with X^0 initial conditions can be written as

$$\tilde{Z}_\epsilon^{(j)}(t) \stackrel{\text{def}}{=} Y^{(j)}(t) + Z_\epsilon^{(j)}(t) \ . \tag{2.48}$$

The process $\tilde{Z}^{(j)}(t,\cdot)$ enjoys, by Proposition A.0.5 and the proprieties of the heat semigroup (B.3)

$$\sup_{0 \le t \le T} t^{(\beta+\nu)\frac{|\mathbf{k}|}{2}} \left\| \tilde{Z}^{:\mathbf{k}:}(t,\cdot) \right\|_{\mathcal{C}^{-\nu}} \le C^* \tag{2.49}$$

where $C^* = C^*(T, \|X_0\|_{\mathcal{C}^{-\nu}}, \beta, \nu, n, \left\|Z^{:\mathbf{k}:}\right\|_{\mathcal{C}^{-\nu}} \text{ for } |\mathbf{k}| \le 2n-1)$. Let

$$\mathfrak{c}_\epsilon = \frac{1}{4} \sum_{0<|\omega|\le\epsilon^{-1}} \frac{1}{\pi^2|\omega|^2}$$

and define the difference

$$A_\epsilon(t) \stackrel{\text{def}}{=} \mathfrak{c}_\varepsilon - \mathfrak{c}_\varepsilon(t) = -\frac{t}{2} + \frac{1}{4} \sum_{0<|\omega|\le\epsilon^{-1}} \frac{e^{-2t\pi^2|\omega|^2}}{\pi^2|\omega|^2} \qquad A(t) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0} A_\epsilon(t)$$

With $A(t) \sim \log(t^{-1})$ for $t \to 0$ and $|A(t)| \sim t$ as $t \to \infty$.
We are now ready to describe the notion of solution to equation (2.46), first defined in [DPD03]. We say that X solves (2.46) if $X(t,\cdot) = \tilde{Z}(t,\cdot) + V(t,\cdot)$ and the process V, solves the PDE

$$\begin{cases} \partial_t V^{(j)}(\cdot,t) & = \Delta V^{(j)}(\cdot,t) + \overline{\Psi}^{(j)}\left(t, (\tilde{Z}^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(V_\gamma(\cdot,t)) \\ V^{(j)}(\cdot,t) & = 0 \end{cases} \tag{2.50}$$

For

$$\overline{\Psi}^{(j)}\left(t, (\tilde{Z}^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(V_\gamma(\cdot,t)) = \mathfrak{p}^{(j)}(\tilde{Z}(\cdot,t) + V(\cdot,t)) \tag{2.51}$$

where

$$\mathfrak{p}^{(j)}(\tilde{Z}(\cdot,t) + V(\cdot,t)) \stackrel{\text{def}}{=} \sum_{|\mathbf{b}|+|\mathbf{a}|\le 2n-1} b^{(j)}_{\mathbf{a},\mathbf{b}}(t) V^{\mathbf{a}}(\cdot,t) \tilde{Z}^{:\mathbf{b}:}(\cdot,t) \tag{2.52}$$

for some coefficients with

$$|b^{(j)}_{\mathbf{a},\mathbf{b}}(t)| \lesssim |A(t)|^{\frac{2n-1-|\mathbf{a}+\mathbf{b}|}{2}} . \tag{2.53}$$

We recall that the products between $\tilde{Z}^{:\mathbf{b}:}(\cdot,t)$ and $V^{\mathbf{a}}(\cdot,t)$ are well defined thanks to Proposition A.0.5 and the fact that $V(t,\cdot) \in \mathcal{C}^\alpha(\mathbb{T}^2, \mathbb{R}^m)$ for any $\alpha < 2$. In particular (2.50) is a PDE that depends on a given realization of the linear process $\tilde{Z}$ and its Wick powers.
The next theorem completes the existence and uniqueness theory behind the limiting equation.

Theorem 2.2.20 *For $0 < \nu < \frac{2}{2n-1}$ small enough and initial data $X^0 \in \mathcal{C}^{-\nu}(\mathbb{T}^2, \mathbb{R}^m)$. For a realization of*

$$\mathbf{z}_{\mathbf{k}} \in L^\infty([0,T]; \mathcal{C}^{-\nu}(\mathbb{T}^2)) \qquad \textit{for } |\mathbf{k}| \le 2n-1 ,$$

let $\mathcal{S}_T$ the solution map that associates to $(\mathbf{z}_{\mathbf{k}})_{|\mathbf{k}|\le 2n-1}$ the solution V to the PDE (2.50).

The solution map exists, it is unique and it is Lipschitz continuous for all $\nu, \kappa > 0$ *with* $\kappa > (2n-1)\nu$ *sufficiently small as*

$$\begin{aligned}\mathcal{S}_T : [L^\infty([0,T];\mathcal{C}^{-\nu}(\mathbb{T}^2))]^{n^*} \mapsto & \mathcal{C}\left([0,T], \mathcal{C}^{2-\nu-\kappa}(\mathbb{T}^2, \mathbb{R}^m)\right) \\ \{\mathbf{z_k}\}_{|\mathbf{k}|\le 2n-1} \to & V\end{aligned}$$

Proof. The same proof in [MW17a, SW16, MW17b] applies to the vector valued problem, see also [TW16, Sec. 3] for some bounds which are independent on the initial conditions.

□

We now spend few words about the existence of the solution for all times. With a general polynomial, the process is expected to have a blowup in finite time. To see this it is sufficient to look at the behaviour, for instance, of the differential equation

$$\dot{x}(t) = x^2(t) \qquad x(0) = 1$$

whose solution $x(t) = (1-t)^{-1}$ diverges at $t = 1$.
In fact, the Assumption M2 guarantees the well posedness of the solution for all times. The proof of this fact for $m = 1$ is presented in [MW17b, Sec. 6], and it consists in providing L^p-bounds for the process $V^{(j)}$ testing $V^{(j)p-1}$ with 2.50 via the assumption on the leading coefficient of the polynomial given in M2. The application to our case is straightforward.
We are now ready to state the main result of this chapter, which will be proved in Section 2.5.

Theorem 2.2.21 *Let* X_γ *the multidimensional process defined from the Glauber dynamic as in Section 2.2.*
For $\nu > 0$ *small enough, let the reference measure* ν_γ *and the initial condition satisfy the assumptions* (I1)*,* (I2')*,* (M0)*,* (M1)*,* (M2) *in Section 2.2.3.*
Then the process X_γ *converges in distribution in* $\mathcal{D}\left([0,T];\mathcal{C}^{-\nu}\right)$ *to the solution, in the sense of Section 2.2.6,* X *of the SPDE in* (2.46).

Recall that by Remark 2.2.15, it is sufficient to work under the condition $\tau_{\gamma,\mathfrak{m}} > T$.
The proof of the main theorem follows exactly as in [MW17a, Theorem 2.1], where the only bound needed is provided by Proposition 2.5.3.
Theorem 2.2.21 implies, for instance the following corollary

Corollary 2.2.22 *Consider the* m*-vector model defined in Remark 2.2.5. Suppose that the law at time zero satisfies* $\mathbb{E}\left\|X_\gamma^0\right\|_{\mathcal{C}^{-\nu}} < \infty$*. Then the Glauber dynamic converges to the solution of* (2.4).

Proof. It is easy to see that the assumptions in Subsection 2.2.3 are satisfied (except the condition $\int_S |\eta|^2 \nu_\gamma(d\eta) = 1$). We can then apply the theorem to $\sigma'_x(t) = \sqrt{m}\sigma_x(t)$ with the new invariant measure $\beta' = \frac{1}{m}\beta$. It is easy to see that the calculation in (2.14) yields

$$(\beta' m)h'_\gamma(x,t) - \frac{1}{m+2}(\beta' m)^3 |h'_\gamma(x,t)|^2 h'_\gamma(x,t) + \mathcal{O}(|h'_\gamma(x,t)|^5)\,.$$

If $\beta' m = 1 + \gamma^2 c_\gamma + o(\gamma^2)$, then Theorem 2.2.21 implies that $X'_\gamma := \delta^{-1} h'_\gamma = \delta^{-1}\sqrt{m} h_\gamma = \sqrt{m} X_\gamma$ converges to

$$\partial_t X' = \Delta X' - \frac{1}{m+2} : |X'|^2 X' : + \xi$$

and therefore the original field converges to the solution of (2.4). □

2.3 The linearized process

In order to prove convergence in law for the Glauber dynamic defined in Section 2.2, we will introduce the linearized dynamic and start proving convergence in law of the linearized dynamic to the solution of the multivariate heat equation. The strategy that we will be using is the same as [MW17a]. We will first show tightness of the linear process and then characterize the law with the martingale problem.

In this section the solution of the discrete linearized dynamic Z_γ is presented. Recall the definition of the Fourier transform and the extension operator in Section 1.2.2. The process Z_γ will be defined over the lattice Λ_ε and it will be an approximation of the solution of the (vector valued) stochastic heat equation for frequencies ω such that $|\omega|_\infty \leq \epsilon^{-1}\gamma$.

It will be convenient to define for a field $Y : \Lambda_\varepsilon \to \mathbb{R}$, its *lower* and *higher truncation*

$$Y^{high} \stackrel{\text{def}}{=} \sum_{2^k \geq \frac{3}{8}\frac{\epsilon^{-1}}{2n-1}} \delta_k Y \qquad Y^{low} \stackrel{\text{def}}{=} \sum_{2^k < \frac{3}{8}\frac{\epsilon^{-1}}{2n-1}} \delta_k Y \tag{2.54}$$

as processes over the continuous torus $\mathbb{T}^2 \to \mathbb{R}$, and analogous definitions can be given in the vector valued processes. The threshold $\frac{3}{8}\frac{1}{2n-1}\epsilon^{-1}$ has been chosen in such a way that the operation of taking the $2n-1$ power of the lower truncation Y^{low} commutes with the extension operator (1.9).

In order to prove the tightness of the laws of the different processes as $\gamma \to 0$ we introduce the approximation $R_{\gamma,t}$ to Z_γ.

2.3.1 Wick powers of the rescaled field

We will write the solution X_γ on Λ_ε as

$$\begin{aligned} X_\gamma(\cdot,t) = P_t^\gamma X_\gamma^0 + \int_0^t P_{t-s}^\gamma K_\gamma *_\epsilon (\mathfrak{p}_\gamma(X_\gamma)(\cdot,s) + E_\gamma(\cdot,s))\, ds \\ + \int_0^t P_{t-s}^\gamma dM_\gamma(\cdot,s) \end{aligned} \tag{2.55}$$

where $\mathfrak{p}_\gamma$ has been defined in (2.22) and $X_\gamma^0 = \delta^{-1}h_\gamma(\cdot,0)$ is the initial condition. We will now denote by $Z_\gamma(x,t) = \left(Z_\gamma^{(1)}(x,t),\dots,Z_\gamma^{(m)}(x,t)\right)$ the mild solution

$$Z_\gamma(x,t) \stackrel{\text{def}}{=} \int_{r=0}^t P_{t-r}^\gamma dM_\gamma(x,s) \tag{2.56}$$

of the approximation of the stochastic heat equation on Λ_ε

$$\begin{cases} dZ_\gamma(x,t) & = \Delta_\gamma Z_\gamma(x,t) + dM_\gamma(x,t) \\ Z_\gamma(x,0) & = 0 \end{cases} \tag{2.57}$$

And we will extend Z_γ to the whole torus $\mathbb{T}^2$, by considering the trigonometric polynomial of degree N that coincides with Z_γ on Λ_ε. Following [MW17a, SW16] we introduce a martingale approximation of Z_γ, defined for $s \le t$ as

$$R_{\gamma,t}(x,s) = \int_{[0,s)} P_{t-r}^\gamma dM_\gamma(x,r)\ . \tag{2.58}$$

From its definition $R_{\gamma,t}$ is a martingale for $0 \le s \le t$ and $\lim_{s\to t} R_{\gamma,t}(x,s) = Z_\gamma(x,t)$ in $\mathcal{C}^{-\kappa}$ for any $\kappa > 0$. We will now define recursively the higher renormalized powers of $R_{\gamma,t}$. Such a definition might not seem intuitive, but it has the advantage of producing automatically a martingale.

We recall that $R_{\gamma,t} = (R_{\gamma,t}^{(1)},\dots,R_{\gamma,t}^{(m)})$ and every renormalized power is indexed by $\mathbf{k} = (k_1,\dots,k_m) \in \mathbb{N}^m$. We then call the degree of the multiindex $\mathbf{k}$ the quantity $|\mathbf{k}| = \sum_{i=1}^m k_i$. To be consistent with the notations, if $|\mathbf{k}| = 1$ we simply consider $R_{\gamma,t}^{\mathbf{k}} \stackrel{\text{def}}{=} R_{\gamma,t}^{(i)}$ if $\mathbf{k}$ is nonzero only in the i-th position.

We then define, for $x \in \Lambda_\varepsilon$ and $0 \le s \le t$

$$R_{\gamma,t}^{:\mathbf{k}:}(x,s) = R_{\gamma,t}^{:k_1,k_2,\dots,k_m:}(x,s) \stackrel{\text{def}}{=} \sum_{i=1}^m k_i \int_{[0,s)} R_{\gamma,t}^{:k_1,\dots,k_i-1,\dots,k_m:}(x,r^-) dR_{\gamma,t}^{(i)}(x,r)\ . \tag{2.59}$$

Where the left limit $R_{\gamma,t}^{:k_1,\dots,k_i-1,\dots,k_n}(x,r^-)$ guarantees that the above quantity is a martin-

gale for all $\mathbf{k}$.
The above definition has the drawback that it is defined only on Λ_ε. In order to extend it to the whole torus $\mathbb{T}^2$, it turns out to be more convenient to work with another definition of $R^{:\mathbf{k}:}_{\gamma,t}$ via the Fourier series

$$\hat{R}^{:\mathbf{k}:}_{\gamma,t}(\omega, s) \stackrel{\text{def}}{=} \sum_{i=1}^{m} k_i \int_{[0,s)} \frac{1}{4} \sum_{\omega' \in \mathbb{Z}^2} \hat{R}^{:k_1,\dots,k_i-1,\dots,k_m:}_{\gamma,t}(\omega - \omega', r^-) d\hat{R}^{(i)}_{\gamma,t}(\omega', r) \ . \tag{2.60}$$

It is immediate to verify that (2.60) defines an extension to $\mathbb{T}^2$ of (2.59) and it is a Fourier polynomial of degree $4|\mathbf{k}|\epsilon^{-2}$.
For multiindex $\mathbf{k}$ and $x \in \mathbb{T}^2$, $0 \leq t$ define

$$Z^{:\mathbf{k}:}_{\gamma}(x, t) \stackrel{\text{def}}{=} \lim_{s \nearrow t} R^{:\mathbf{k}:}_{\gamma,t}(x, s) \ . \tag{2.61}$$

As the notation suggests, the quantities $Z^{:\mathbf{k}:}_{\gamma}(\cdot, t)$ are going to be an approximation for the Wick powers of the solution of the linearized process. This relation will be made more precise in Proposition 2.4.2 in the next section.

The rest of the section is devoted to show that the processes $R^{:\mathbf{k}:}_{\gamma,t}(\cdot, t)$ belong to $\mathcal{C}([0,T], \mathcal{C}^{-\nu})$ for any small $\nu > 0$, which is the content of Proposition 2.3.4.
Using (2.18) the quadratic covariation of (2.58) is given by

$$\begin{aligned}
&\left\langle R^{(i)}_{\gamma,t}(z_1, \cdot), R^{(j)}_{\gamma,t}(z_2, \cdot) \right\rangle_s \\
&= \int_{[0,s)} \sum_{y_1, y_2 \in \Lambda_\varepsilon} \epsilon^{2d} P^{\gamma}_{t-r}(z_1 - y_1) P^{\gamma}_{t-r}(z_2 - y_2) d \left\langle M^{(i)}_{\gamma,\cdot}(y_1), M^{(j)}_{\gamma,\cdot}(y_2) \right\rangle_r \\
&= \sum_{z \in \Lambda_\varepsilon} \epsilon^d \int_{[0,s)} P^{\gamma}_{t-r} K_\gamma(z - y)^2 Q^{i,j}_{\mathfrak{m}}(r, z) dr \ .
\end{aligned}$$

By Proposition 2.2.17, the above expectation is bounded by

$$\left(\mathbb{E} \left| \left\langle R^{(i)}_{\gamma,t}(y, \cdot), R^{(j)}_{\gamma,t}(y, \cdot) \right\rangle_s \right|^q \right)^{1/q} \leq C(\mathfrak{m}, q) \sum_{z \in \Lambda_\varepsilon} \epsilon^2 \int_{[0,s)} P^{\gamma}_{t-r} K_\gamma(z - y)^2 dr \tag{2.62}$$

and therefore, for $s < t$, $R_{\gamma,t}(y, s)$ is a true martingale.
We expect to get the orthogonality of the martingales for $i \neq j$ in the limit as $\gamma \to 0$.

The next estimate is needed in Proposition 2.3.4 to control the norm of the iterated integrals of the process $R^{:\mathbf{k}:}_{\gamma,t}(\cdot, s)$. This is essentially lemma 4.1 of [MW17a] for a particular choice of the kernels. We will provide a proof of it since it is a key estimate, even though the proof follows closely the one in [MW17a], with the only difference that in our case an

Hölder inequality has been used to deal with the fact that the spins in our model are not bounded uniformly by 1. Furthermore, the result is not stated in its more general form in order to avoid the introduction of notations that are not going to be used in the rest of the paper.
For the next proposition we will use the notation $R^{:\mathbf{k}:}_{\gamma,t}(\varphi, s)$ to denote the $L^2(\Lambda_\varepsilon)$ scalar product between $R^{:\mathbf{k}:}_{\gamma,t}(\cdot, s)$ and a test function φ.

Proposition 2.3.1 *Let $\varphi : \mathbb{T}^2 \to \mathbb{R}$ be a smooth test function, let $p > 2$ and $\kappa > 0$, then there exists a constant $C = C(\mathbf{k}, p, \mathfrak{m}, \kappa)$, such that*

$$\left(\mathbb{E} \sup_{0\le s\le t} |R^{:\mathbf{k}:}_{\gamma,t}(\varphi, s)|^p\right)^{\frac{2}{p}}$$
$$\le C\sum_{i=1}^{m} k_i^2 \int_{r=0}^{t} \sum_{y\in\Lambda_\varepsilon} \epsilon^2 \mathbb{E}\left[\left|R^{:k_1,\dots,k_i-1,\dots,k_m:}_{\gamma,t}(P^\gamma_{t-r}K_\gamma(\cdot - y)\varphi, r^-)\right|^{p+\kappa}\right]^{\frac{2}{p+\kappa}} dr$$
$$+ C(\delta^{-1}\epsilon^2)^{2-\kappa} \sum_{i=1}^{m} k_i\, \mathbb{E}\left[\sup_{r\le t}\sup_{y\in\Lambda_\varepsilon}\left|R^{:k_1,\dots,k_i-1,\dots,k_m:}_{\gamma,t}(P^\gamma_{t-r}K_\gamma(\cdot - y)\varphi, r^-)\right|^{p+\kappa}\right]^{\frac{2}{p+\kappa}}$$

and reiterating the above formula we obtain

$$\left(\mathbb{E} \sup_{0\le s\le t} |R^{:\mathbf{k}:}_{\gamma,t}(\varphi, s)|^p\right)^{\frac{2}{p}}$$
$$\le C\int_{r_1=0}^{t}\int_{r_2=0}^{r_1}\cdots\int_{r_{|\mathbf{k}|}=0}^{r_{|\mathbf{k}|-1}} \sum_{\bar y\in(\Lambda_\varepsilon)^{|\mathbf{k}|}} \epsilon^{2|\mathbf{k}|}\left\langle \varphi, F^t_{|\mathbf{k}|}(\bar y, \bar r)\right\rangle^2_{L^2(\Lambda_\varepsilon)} d\bar r + \mathbf{err} \quad (2.63)$$

where

$$F^t_l(y_1,\dots,y_l,r_1,\cdots,r_l)(x) = \prod_{i=1}^{l} P^\gamma_{t-r_i}K_\gamma(x-y_i) \quad (2.64)$$

and the error term is bounded by

$$\mathbf{err} \lesssim (\delta^{-1}\epsilon^2)^{2-\kappa} \sum_{l=1,\dots,|\mathbf{k}|}\int_{r_1=0}^{t}\int_{r_2=0}^{r_1}\cdots\int_{r_{l-1}=0}^{r_{l-2}} \sum_{y_1,\dots,y_{l-1}\in\Lambda_\varepsilon} \epsilon^{2(l-1)}$$
$$\times \sup_{\mathbf{a}\in\mathbb{N}^m:|\mathbf{a}|=l} \mathbb{E}\left[\sup_{\substack{r_l<r_{l-1}\\ y_l\in\Lambda_\varepsilon}}\left|R^{:\mathbf{k}-\mathbf{a}:}_{\gamma,t}\left(\varphi F^t_l(y_1,\dots,y_l,r_1,\dots,r_l), r_l\right)\right|^{p+l\kappa}\right]^{\frac{2}{p+l\kappa}} dr_1\cdots dr_{l-1}. \quad (2.65)$$

Proof. It is easy to see that the above formula holds for $|\mathbf{k}| = 1$ and any $p > 2$. We then use the Burkholder-Davis-Gundy inequality and the induction on $|\mathbf{k}|$ to prove that it holds also

for any $\mathbf{k} \in \mathbb{N}^m$.
From the recursive formula (2.59) we compute the quadratic variation of

$$R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(\varphi,s) = \sum_{i=1}^{m} k_i \int_{[0,s)} \sum_{x\in\Lambda_\varepsilon} \epsilon^2 \varphi(x) R_{\gamma,t}^{:k_1,\ldots,k_i-1,\ldots,k_m:}(x,r^-) dR_{\gamma,t}^{(i)}(x,r) \ .$$

In order to apply the Burkholder-Davis-Gundy inequality we have to estimate the quadratic variation of the process and the size of the jumps.
The quadratic variation of the process is then

$$\begin{aligned}
&\left\langle \sum_{x\in\Lambda_\varepsilon} \epsilon^2 \varphi(x) R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(x,\cdot) \right\rangle_s \leq \\
&C(\mathbf{k}) \sum_i k_i^2 \int_{[0,s)} \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 \varphi(x)\varphi(y) \\
&\times R_{\gamma,t}^{:k_1,\ldots,k_i-1,\ldots,k_m:}(x,r^-) R_{\gamma,t}^{:k_1,\ldots,k_i-1,\ldots,k_m:}(y,r^-) d\left\langle R_{\gamma,t}^{(i)}(x,\cdot), R_{\gamma,t}^{(i)}(y,\cdot) \right\rangle_r \\
&\lesssim \sum_i k_i^2 \int_{[0,s)} \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |R_{\gamma,t}^{:k_1,\ldots,k_i-1,\ldots,k_m:}(\varphi(\cdot)P_{t-r}^{\gamma}K_\gamma(\cdot - z), r^-)|^2 Q_{\mathfrak{m}}^{i,i}(r^-,z) dr \ .
\end{aligned}$$

We define the jump of a cadlag process at time $r \in \mathbb{R}$, as $\Delta_r R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(\varphi,r)$ and it is given by

$$\epsilon^2 \delta^{-1} \sum_i k_i \sup_{\substack{z\in\Lambda_\varepsilon \\ 0\leq r\leq s}} \left| R_{\gamma,t}^{:k_1,\ldots,k_i-1,\ldots,k_m:}(\varphi(\cdot)P_{t-r}^{\gamma}K_\gamma(\cdot - z), r^-) \right| |\Delta_r \sigma_z(\alpha^{-1}r)| \tag{2.66}$$

where $\Delta_r \sigma_z(\alpha^{-1}r) = \sigma_z(\alpha^{-1}r) - \sigma_z(\alpha^{-1}r^-)$. Therefore we have that

$$\begin{aligned}
&\left(\mathbb{E} \sup_{0\leq s\leq t} |R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(\varphi,s)|^p \right)^{\frac{2}{p}} \\
&\leq C(p) \left(\mathbb{E} \left\langle R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(\varphi,\cdot) \right\rangle_t^{\frac{p}{2}} \right)^{\frac{2}{p}} + C(p) \left(\mathbb{E} \sup_{0\leq s\leq t} \left| \Delta_r R_{\gamma,t}^{:k_1,k_2,\ldots,k_m:}(\varphi,r) \right|^p \right)^{\frac{2}{p}}
\end{aligned} \tag{2.67}$$

Then use Minkowski's inequality with exponent $p/2 > 1$

$$\begin{aligned}
&\left(\mathbb{E}\left\langle R^{:k_1,k_2,\ldots,k_m:}_{\gamma,t}(\varphi,\cdot)\right\rangle_s^{\frac{p}{2}}\right)^{\frac{2}{p}} \\
&\qquad \leq C(\mathbf{k},p)\sum_i k_i^2 \int_{[0,s)} \sum_{z\in\Lambda_\varepsilon} \epsilon^2 \times \\
&\qquad \times \mathbb{E}\left[\left(|R^{:k_1,\ldots,k_i-1,\ldots,k_m:}_{\gamma,t}(\varphi(\cdot)P^\gamma_{t-r}K_\gamma(\cdot - z), r^-)|^2 Q^{i,i}_{\mathfrak{m}}(r^-,z)\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} dr
\end{aligned}$$

and at this point we use the Hölder inequality to separate the term $Q^{i,i}_{\mathfrak{m}}(r^-,z)$

$$\begin{aligned}
&\mathbb{E}\left[\left(|R^{:k_1,\ldots,k_i-1,\ldots,k_m:}_{\gamma,t}(\varphi(\cdot)P^\gamma_{t-r}K_\gamma(\cdot - z), r^-)|^2 Q^{i,i}_{\mathfrak{m}}(r^-,z)\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} \\
&\qquad\qquad \leq C(p,\kappa)\mathbb{E}\left[\left|R^{:k_1,\ldots,k_i-1,\ldots,k_m:}_{\gamma,t}(\varphi(\cdot)P^\gamma_{t-r}K_\gamma(\cdot - z), r^-)\right|^{p+\kappa}\right]^{\frac{2}{p+\kappa}}
\end{aligned}$$

Where in the last line we used the bounds over the moments of $Q^{i,i}_{\mathfrak{m}}(r^-,z)$ provided in Proposition 2.2.17. This is the only difference with the proof of [MW17a], where a uniform bound on $Q^{i,i}_{\mathfrak{m}}(r^-,z)$ is available. We can then use induction on the integrand, with the new test function $\varphi(\cdot)P^\gamma_{t-r}K_\gamma(\cdot - z)$.
We now bound the jump part inside the summation in (2.66) with Lemma B.0.3

$$|\delta^{-1}\epsilon^2 P^\gamma_{t-r}K_\gamma(x-z)| \leq \delta^{-1}\gamma^2\log(\gamma^{-1})$$

and using the Hölder inequality (considering $\frac{\kappa}{\kappa+p} + \frac{p}{p+\kappa} = 1$) together with

$$\left(\mathbb{E}\sup_{0\leq r\leq t, z\in\Lambda_\varepsilon} |\Delta_{r^-}\sigma(z)|^{p+\frac{p^2}{\kappa}}\right)^{\frac{\kappa}{\kappa+p}} \leq \left(\mathbb{E}\sum_{z\in\Lambda_\varepsilon}\sum_{0\leq r\leq t}|\sigma_z(\alpha^{-1}r)|^{q(p+\frac{p^2}{\kappa})}\right)^{\frac{\kappa}{(\kappa+p)q}} \tag{2.68}$$

where the last sum is over all jumps that happened at site z in $[0,t]$. Since the number of jumps is a Poisson process with intensity bounded by α^{-1}, the last expectation can be replaced by

$$\mathbb{E}\sum_{z\in\Lambda_\varepsilon}\sum_{0\leq r\leq t}|\sigma_z(\alpha^{-1}r)|^{q(p+\frac{p^2}{\kappa})} = \alpha^{-1}\mathbb{E}\sum_{z\in\Lambda_\varepsilon}\int_0^t |\sigma_z(\alpha^{-1}r)|^{q(p+\frac{p^2}{\kappa})}dr \leq C\epsilon^{-2}\alpha^{-1}$$

and if we choose q large enough we have that the last line in (2.67) is bounded by

$$C(q,p,\kappa,\mathfrak{m})(\epsilon^2\delta^{-1})^2(\epsilon^2\alpha^1)^{-\frac{2\kappa}{q(p+\kappa)}}$$
$$\times\mathbb{E}\left[\sup_{\substack{z\in\Lambda_\varepsilon\\0\le r\le s}}\left|\sum_{x\in\Lambda_\varepsilon}\epsilon^2\varphi(x)P^\gamma_{t-r}K_\gamma(x-z)R^{:k_1,\ldots,k_i-1,\ldots,k_m:}_{\gamma,t}(x,r^-)\right|^{p+\kappa}\right]^{\frac{2}{p+\kappa}}$$

where for q large and by (2.20) we can assume $(\epsilon^2\alpha)^{-\frac{\kappa}{q(p+\kappa)}}\ll(\epsilon^2\delta^{-1})^{-\kappa}$. This proves the inductive step. The rest of the estimates follows directly from the proof in [MW17a]. $\square$

Remark 2.3.2 In the above proposition the regularity of φ is not entering into the proof, hence it is easy to see that one could take as φ the discrete Dirac delta on the lattice and, using Lemma B.0.4, obtain the bound

$$\mathbb{E}[\sup_{0\le s\le t}|R^{:\mathbf{k}:}_{\gamma,t}(x,s)|^p]\lesssim\log^{p|\mathbf{k}|}(\gamma^{-1})\ .$$

A result similar to the one in Proposition 2.3.1 can be proven also for

$$\mathbb{E}\sup_{0\le r<t}|R^{:\mathbf{k}:}_{\gamma,t}(\varphi,r)-R^{:\mathbf{k}:}_{\gamma,t}(\varphi,s\wedge r)|^p\qquad\mathbb{E}\sup_{0\le r<t}|R^{:\mathbf{k}:}_{\gamma,t}(\varphi,r)-R^{:\mathbf{k}:}_{\gamma,s}(\varphi,s\wedge r)|^p\ .$$

Since the proof is exactly the same as in the case of Proposition 2.3.1, we only state the result.

Corollary 2.3.3 *Under the same assumptions as Proposition 2.3.1 and the definition of F^t_l given in* (2.64) *we have*

$$\begin{aligned}&\left(\mathbb{E}\sup_{0\le r<t}|R^{:\mathbf{k}:}_{\gamma,t}(\varphi,r)-R^{:\mathbf{k}:}_{\gamma,t}(\varphi,s\wedge r)|^p\right)^{\frac{2}{p}}\\&\quad\le C\int_{r_1=s}^t\int_{r_2=0}^{r_1}\cdots\int_{r_{|\mathbf{k}|}=0}^{r_{|\mathbf{k}|-1}}\sum_{\bar y\in(\Lambda_\varepsilon)^{|\mathbf{k}|}}\epsilon^{2|\mathbf{k}|}\left\langle\varphi,F^t_{|\mathbf{k}|}(\bar y,\bar r)\right\rangle^2_{L^2(\Lambda_\varepsilon)}d\bar r+\mathbf{err}_1\end{aligned}\tag{2.69}$$

$$\begin{aligned}&\left(\mathbb{E}\sup_{0\le r<s}|R^{:\mathbf{k}:}_{\gamma,t}(\varphi,r)-R^{:\mathbf{k}:}_{\gamma,s}(\varphi,r)|^p\right)^{\frac{2}{p}}\\&\le C\int_{r_1=0}^s\int_{r_2=0}^{r_1}\cdots\int_{r_{|\mathbf{k}|}=0}^{r_{|\mathbf{k}|-1}}\sum_{\bar y\in(\Lambda_\varepsilon)^{|\mathbf{k}|}}\epsilon^{2|\mathbf{k}|}\left\langle\varphi,F^t_{|\mathbf{k}|}(\bar y,\bar r)-F^s_{|\mathbf{k}|}(\bar y,\bar r)\right\rangle^2_{L^2(\Lambda_\varepsilon)}d\bar r+\mathbf{err}_2\end{aligned}\tag{2.70}$$

and the error terms have the same form of (2.65) *with the replacement of the kernel F^t_l precisely as done in* (2.69) *and* (2.70).

With the above considerations we are ready to state the bounds on the solution of the linear dynamic and its Wick powers.

Proposition 2.3.4 *There exists $\gamma_0 > 0$ such that the following holds. For every multiindex $\mathbf{k} \in \mathbb{N}^n$, $p > 1$ $\nu > 0$, $T \geq 0$, $0 \leq \lambda < \frac{1}{2}$ and $0 < \kappa \leq 1$, there exists a constant $C = C(\mathbf{k}, , \nu, T, \lambda, \kappa)$ such that $0 \leq s \leq t \leq T$ and $0 < \gamma < \gamma_0$*

$$\begin{aligned}
\mathbb{E} \sup_{0\leq r\leq t} \left\| R^{:\mathbf{k}:}_{\gamma,t}(\cdot, r) \right\|^p_{\mathcal{C}^{-\nu-2\lambda}} &\leq Ct^{\lambda p} + C\gamma^{p(1-\kappa)} \\
\mathbb{E} \sup_{0\leq r\leq t} \left\| R^{:\mathbf{k}:}_{\gamma,t}(\cdot, r) - R^{:\mathbf{k}:}_{\gamma,s}(\cdot, r\wedge s) \right\|^p_{\mathcal{C}^{-\nu-2\lambda}} &\leq C|t-s|^{\lambda p} + C\gamma^{p(1-\kappa)} \\
\mathbb{E} \sup_{0\leq r\leq t} \left\| R^{:\mathbf{k}:}_{\gamma,t}(\cdot, r) - R^{:\mathbf{k}:}_{\gamma,t}(\cdot, r\wedge s) \right\|^p_{\mathcal{C}^{-\nu-2\lambda}} &\leq C|t-s|^{\lambda p} + C\gamma^{p(1-\kappa)}
\end{aligned}$$

And, taking the limit $s \to t$ of the martingales, the same bounds are satisfied by Z_γ.

Proof. The proof is equal to [MW17a, Prop 4.2], with the use of the bounds (2.63), (2.69), (2.70). We will only need to apply Proposition 2.3.1 repeatedly with φ equals to the kernel of every Paley-Littlewood projection. □

Remark 2.3.5 For the above estimates we didn't use the fact that, for $i \neq j$, the martingales $M^{(i)}_t$ and $M^{(j)}_t$ are orthogonal in the limit. The calculation of the covariation will be addressed in the next section.

Remark 2.3.6 As the bounds in Proposition 2.3.4 for $R^{:\mathbf{k}:}_{\gamma,t}(s, \cdot)$ are uniform in s, the same bounds are available for the process Z_γ defined in (2.56).

We now state a lemma that gives a better control over the high frequencies of the fluctuation field, which will be used when we extend the powers of the linear process to the continuous torus in Section 2.5. The next lemma correspond to [MW17a, Lemma 4.6], and the proof follows exactly the same steps.

Lemma 2.3.7 *Recall the definition of Z^{high}_γ in (2.54). For all $p \geq 1$, $\kappa > 0$ and $T > 0$, there exists a constant $C = C(p, \kappa, T, \mathfrak{m})$ such that for all $\gamma < \gamma_0$ and $0 \leq t \leq T$*

$$\mathbb{E}\left[\left\| Z^{\mathrm{high}}_\gamma(\cdot, t) \right\|^p_{L^\infty(\mathbb{T}^2)} \right]^{1/p} \leq C\gamma^{1-\kappa} . \tag{2.71}$$

2.4 Tightness and convergence for the linearized system

In this section we state the tightness result for the powers of the linearized process Z_γ given by (2.61) and we will characterize the limit with a martingale problem in Subsection 2.4.1.

This is the main reason for the introduction of the stopping time in Subsection 2.2.5. For a separable metric space $\mathcal{A}$, denote with $\mathcal{D}([0,T],\mathcal{A})$ the Skorokhod space of cadlag function taking value in $\mathcal{A}$ endowed with the Skorokhod topology: this makes $\mathcal{D}([0,T],\mathcal{A})$ a metric space as well with the distance

$$\mathrm{dist}_{\mathcal{D}(\mathbb{R}^+,\mathcal{A})} = \sup_{\lambda\in\Lambda_{[0,T]}} \max\left\{|\lambda - id|_\infty, |f\circ\lambda - g|_{L^\infty[0,T]}\right\}$$

for $f,g\in\mathcal{D}(\mathbb{R}^+,\mathcal{A})$ and for $\lambda\in\Lambda_{[0,T]}$ the space of continuous reparametrization of the interval $[0,T]$.

The following proposition corresponds to [MW17a, Proposition 5.4] and provides the tightness result for the laws of the Wick powers. We recall that Z_γ is a multivariate process with m components and $Z_\gamma^{:\mathbf{k}:}(t,\cdot)\in\mathcal{C}^{-\nu}(\mathbb{T}^d)$ by Proposition 2.3.4 and Remark 2.3.6.

Proposition 2.4.1 *Denote by γ_0 the constant in [MW17a, Lemma 8.2]. For any multiindex $\mathbf{k}\in\mathbb{N}^n$ and $\nu>0$, the family $\left\{Z_\gamma^{:\mathbf{k}:};\gamma\in(0,\gamma_0)\right\}$ is tight in $\mathcal{D}(\mathbb{R}^+,\mathcal{C}^{-\nu}(\mathbb{T}^d))$.*
Any weak limit is supported on $\mathcal{C}(\mathbb{R}^+,\mathcal{C}^{-\nu}(\mathbb{T}^d))$ and

$$\sup_{\gamma\in(0,\gamma_0)} \mathbb{E}\sup_{0\le t\le T}\left\|Z_\gamma^{:\mathbf{k}:}(t,\cdot)\right\|_{\mathcal{C}^{-\nu}}^p < \infty\,. \tag{2.72}$$

Proof. The proof is the same as the proof of [MW17a, Proposition 5.4], and it is a consequence of Proposition 2.3.4. □

We will now formalize the fact that the iterated integrals, introduced in Section 2.3, are a convenient approximation of the Wick power of the solution to the linear model. The proof of the next theorems are essentially the same as in [MW17a].

Let $H_{\mathbf{k}}$ be the generalized Hermite polynomial defined in Section 2.2.2, and

$$[R_{\gamma,t}(\cdot,x)]_s \;=\; \left([R_{\gamma,t}^{(i)}, R_{\gamma,t}^{(j)}(\cdot,x)]\right)_{i,j=1}^m$$

the optional quadratic variation matrix. If we define the error

$$E_{\gamma,t}^{:\mathbf{k}:}(s,x) \stackrel{\text{def}}{=} H_{\mathbf{k}}\left(R_{\gamma,t}(s,x),[R_{\gamma,t}(\cdot,x)]_s\right) - R_{\gamma,t}^{:\mathbf{k}:}(s,x)\,, \tag{2.73}$$

then we can prove the following version of [MW17a, Proposition 5.3].

Proposition 2.4.2 *For any multiindex $\mathbf{k} \in \mathbb{N}^m$, $\kappa > 0$, $t > 0$ and $1 \leq p < \infty$ there exists $C = C(\mathbf{k}, p, t, \kappa, \mathfrak{m})$ such that for all $\gamma \in (0, \gamma_0)$*

$$\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{0 \leq s \leq t} |E_{\gamma,t}^{:\mathbf{k}:}(s,x)|^p \leq C\gamma^{p(1-\kappa)} . \tag{2.74}$$

The form of the error considered in (2.73) is somehow unsatisfactory because of the presence of the optional quadratic variation in (2.73). It is possible to prove, however, that the quadratic variation can be approximated by a diagonal matrix. This will be the content of Propositions 2.4.4 and 2.4.7.

The next lemma shows that the quadratic variation $\langle R_{t,\gamma}(\cdot,x)\rangle_s$ approximates the bracket process $[R_{t,\gamma}(\cdot,x)]_s$ as in [MW17a, Lemma 5.1]. Its proof is postponed to Subsection 2.4.2.

Lemma 2.4.3 *Let $x \in \Lambda_N$, $s \in [0,t]$, and define the $(m \times m)$ martingale $U_{\gamma,t}(s,x)$ as*

$$U_{\gamma,t}^{(i,j)}(s,x) \overset{def}{=} \left[R_{t,\gamma}^{(i)}(\cdot,x), R_{t,\gamma}^{(j)}(\cdot,x)\right]_s - \left\langle R_{t,\gamma}^{(i)}(\cdot,x), R_{t,\gamma}^{(j)}(\cdot,x)\right\rangle_s \tag{2.75}$$

for $1 \leq i,j \leq m$.
For all $n \in \mathbb{N}^+$, $t > 0$, $\kappa > 0$ and $p \in [1,\infty]$, there exists a constant $C = C(t,\kappa,p,m)$ such that for $\gamma \in (0,\gamma_0)$

$$\mathbb{E} \sup_{x \in \Lambda_\varepsilon} \sup_{0 \leq s \leq t} |U_{\gamma,t}(s,x)|^p_{m \times m} \leq C\gamma^{p(1-\kappa)} \tag{2.76}$$

where $|\cdot|_{m\times m}$ is the norm in $\mathbb{R}^{m\times m}$ and γ_0 is the constant in [MW17a, Lemma 8.2].

It is clear that the bound in (2.76) can be computed componentwise, and the unidimensional case is proven in [MW17a], where their only assumption used is the boundedness of the rate function of the jumps. It is sufficient in particular to prove it for the diagonal elements.

As a difference with the main reference [MW17a], we now propose a bound on the quadratic variation $[R_{\gamma,t}^{(i)}(\cdot,x), R_{\gamma,t}^{(j)}(\cdot,x)]_s$ that shows that each component of $R_{\gamma,t}$ is asymptotically uncorrelated with the others. Lemma 2.4.3 shows that it is sufficient to bound the predictable quadratic variation. We first define a new approximation of the diverging constant

$$
\begin{aligned}
\mathfrak{c}_{\gamma,t}(s) &= 2\int_0^s \left\|P^\gamma_{t-r}\right\|^2_{L^2(\Lambda_\varepsilon)}dr \\
&= \frac{s}{2} + \sum_{\substack{\omega\in\mathbb{Z}^2\\ 0<|\omega|\le\epsilon^{-1}}} \frac{|\hat{K_\gamma}(\omega)|^2 e^{-2|t-s|\epsilon^{-2}\gamma^2(1-\hat{K_\gamma}(\omega))}}{4\epsilon^{-2}\gamma^2(1-\hat{K_\gamma}(\omega))}\left(1-e^{-2s\epsilon^{-2}\gamma^2(1-\hat{K_\gamma}(\omega))}\right) . \qquad (2.77)
\end{aligned}
$$

The next proposition, whose proof is postponed to the following subsection, is the key estimate behind Proposition 2.4.7.

Proposition 2.4.4 *For $1\le i,j\le m$, $b\in[0,1]$, $\gamma\in(0,\gamma_0)$ and $p\ge 1$ we have*

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{x\in\Lambda_\varepsilon}\sup_{0\le s\le t}\left|\left\langle R^{(i)}_{\gamma,t}(\cdot,x),R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_s-\mathfrak{c}_{\gamma,t}(s)\delta_{i,j}\right|^p\right]^{1/p} \\
&\le C(\kappa,T,\mathfrak{m},\nu,p)\gamma^{1-\nu-\kappa}+C(p,b,\kappa)(1\wedge t^{-b}\alpha^b)\gamma^{-\kappa}
\end{aligned} \qquad (2.78)
$$

Remark 2.4.5 The main difference between Proposition 2.4.4 and [SW16, Proposition 3.4] is that in the latter the bounds on the error gets worse as p grows, while in (2.78) the power of γ in the right-hand-side doesn't depend on p. Such a result is more convenient in our case when the degree of the Wick polynomial is arbitrary large, since the errors containing the renormalization constants diverge as a arbitrarily large power of $\log(\gamma^{-1})$.

With the same proof it is possible to show the following lemma.

Lemma 2.4.6 *For $0\le i,j\le m$, $\kappa>0$, $\nu>0$, and $p>1$, $0\le t\le T$ and for $\phi\in\mathcal{C}\left(\mathbb{T}^2\times[0,T],\mathbb{R}^m\right)$ there exists a constant $C=C(\mathfrak{m},\kappa,\nu,p,T)$*

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\sum_{z\in\Lambda_\varepsilon}\phi^{(i)}(z,s)\phi^{(j)}(z,s)Q^{i,j}_{\mathfrak{m}}(s^-,z)ds-2\delta_{i,j}\int_0^t\left\langle\phi^{(i)}(\cdot,s),\phi^{(j)}(\cdot,s)\right\rangle_{\mathbb{T}^2}ds\right|^p\right] \\
&\le C\gamma^{1-\kappa-\nu}\int_0^t\left\|\phi^{(i)}(s)\phi^{(j)}(s)\right\|^p_{L^p(\mathbb{T}^2)}ds+C\alpha\mathbb{E}[\|\sigma(0)\|^2_{L^2(\Lambda_\varepsilon)}]\left\|\phi^{(i)}\phi^{(j)}\right\|^p_{L^\infty(\mathbb{T}^2\times[0,T])}
\end{aligned} \qquad (2.79)
$$

We conclude the section with a proposition that simplifies the expressions for the Hermite polynomial approximating the iterated integrals $R_{\gamma,t}$, with the replacement of the covariance of the process $R_{\gamma,t}$, with its limiting value. We recall the definition of the constant introduced in (2.28) and we define the values of $\mathfrak{c}_\gamma(t)$, for future reference

$$\mathfrak{c}_\gamma(t) = \frac{t}{2} + \sum_{\substack{\omega\in\mathbb{Z}^2 \\ 0<|\omega|\le\epsilon^{-1}}} \frac{|\hat{K_\gamma}(\omega)|^2}{4\epsilon^{-2}\gamma^2(1-\hat{K_\gamma}(\omega))}\left(1-e^{-2t\epsilon^{-2}\gamma^2(1-\hat{K_\gamma}(\omega))}\right) . \tag{2.80}$$

Together with Proposition 2.4.2, we have the main result of the section:

Proposition 2.4.7 *For a multiindex* $\mathbf{k}\in\mathbb{N}^m$*,* $\kappa>0$*,* $\nu>0, p\ge 1$ *and* $b\in[0,1]$*, under the assumption* (I2')

$$\begin{aligned}&\mathbb{E}\left[\sup_{z\in\Lambda_\varepsilon}\sup_{0\le s\le t}\left|H_{\mathbf{k}}(R_{\gamma,t}(s,z),\mathfrak{c}_{\gamma,t}(s)I_m) - R^{:\mathbf{k}:}_{\gamma,t}(s,z)\right|^p\right]^{\frac{1}{p}} \\ &\le C(\kappa,T,\mathfrak{m},\nu,\mathbf{k},b)\left(\gamma^{(1-\nu-\kappa)}+\gamma^{-\kappa}(1\wedge t^{-b}\alpha^b)\right)\end{aligned} . \tag{2.81}$$

The proof of the above proposition is given at the end of Subsection 2.4.2.

2.4.1 Convergence of the linearized dynamic

Recall the definition of Z_γ given in Subsection 2.2.6 and the tightness of their laws proved in Proposition 2.4.1. For $\nu>0$, we assume in this subsection, that the limit $\gamma\to 0$ is taken along to a fixed converging subsequence of Z_γ.
In this section we show that any limit law solves a martingale problem, more precisely we will use the fact, that the law of the stochastic heat equation is the only solution of a martingale problem (see [MW17a, Appendix C]).
The next result has been proven in [MW17a, Theorem 6.1], the extension to vector-valued processes being straightforward.

Theorem 2.4.8 *Let* $\nu>0$*, The law of the processes* Z_γ *as* $\gamma\to 0$*, converge to the law of* Z*, the solution of the multivariate stochastic heat equation*

$$\begin{cases}\partial_t Z_t &= \Delta Z_t + \sqrt{2}dW_t \\ Z_0 &\equiv 0\end{cases} \tag{2.82}$$

in the topology of $\mathcal{D}\left([0,T];\mathcal{C}^{-\nu}(\mathbb{T}^2,\mathbb{R}^m)\right)$.
Here W *is a* n*-component noise* $(W^{(1)},\dots,W^{(n)})$ *and each of the components is an independent space time white noise on* $L^2([0,T]\times\mathbb{T}^d)$.

The proof is identical to [MW17a, Theorem 6.1]. The only new part is the estimation of the quadratic covariation via Lemma 2.4.6 and the assumption (I2').

We are now ready to state the main result of the section.

Theorem 2.4.9 *For any* $\mathfrak{m}$ *and* $k \geq 1$*, the processes* $(Z_\gamma^{\mathbf{k}})_{|\mathbf{k}|\leq k}$ *converge jointly in law to* $(Z^{\mathbf{k}})_{|\mathbf{k}|\leq k}$ *in the topology of* $\mathcal{D}(\mathbb{R}^+, (\mathcal{C}^{-\nu})^K)$ *where* $K = \binom{k+m-1}{m-1}$

The proof of the above theorem is essentially the same as the proof of theorem 6.2 in [MW17a] or proposition 4.5 in [SW16], and it is based on the approximation $R_{\gamma,t}$ of Z_γ and Proposition 2.4.7.

2.4.2 Proofs of the statements

The aim of this section is to show that the iterated integrals of the process $R_{\gamma,t}(s,x)$ are a good approximation for the Hermite polynomial.

Lemma 2.4.10 *Recall that the Glauber dynamic is stopped as prescribed in Subsection 2.2.5. For any* $p \geq 1$*,* $x \in \Lambda_\varepsilon$ *and* $\kappa > 0$

$$\mathbb{E}\left[\sup_{0\leq r\leq t}\sup_{x\in\Lambda_\varepsilon}|\Delta_r R^{(j)}_{\gamma,t}(\cdot,x)|^p\right]^{1/p} \leq C(p,\kappa,t)\gamma^{1-\kappa}\ . \tag{2.83}$$

Proof. By monotonicity of L^p norms it is sufficient to prove the bound for high values of p. If at microscopic time r, a jump happens at macroscopic site $y \in \Lambda_\varepsilon$, the size of the jump of $R_{\gamma,t}(r,x)$

$$|\Delta_r R_{\gamma,t}(r,x)| = \delta^{-1}\epsilon^2 |P^\gamma_{t-r}K_\gamma(y-x)||\sigma_r(\epsilon^{-1}y) - \sigma_{r^-}(\epsilon^{-1}y)|\ . \tag{2.84}$$

From the form of ν_γ in Subsection 2.2.5,

$$\mathbb{E}|\sigma_r(\epsilon^{-1}y) - \sigma_{r^-}(\epsilon^{-1}y)|^p \leq C(p)(\mathbb{E}|\sigma_r(\epsilon^{-1}y)|^p + \mathbb{E}|\sigma_{r^-}(\epsilon^{-1}y)|^p) < 2C(p,\mathfrak{m})$$

bounded uniformly in the dynamic. Using the fact that, on Λ_ε, $\|P^\gamma_t\|_{L^1(\Lambda_\varepsilon)} = 1$ and $\sup_{x\in\Lambda_\varepsilon} K_\gamma(x) \lesssim \epsilon^{-2}\gamma^2$ one has the following

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{\substack{(r,x)\in[0,t]\times\Lambda_\varepsilon\\ \text{jump at } (r,x)}} |\Delta_r R_{\gamma,t}(r,x)|^p\Big] \\
&\qquad \leq \mathbb{E}\Big[\sum_{i=0}^N \mathbb{E}\Big(|\Delta_{r_i} R_{\gamma,t}(r_i,x_i)|^p\Big|\text{jumps at } (r_i,x_i) : i = 1\dots N\Big)\Big] \\
&\qquad \leq C(p,\mathfrak{m})\delta^{-p}\gamma^{2p}\mathbb{E}\left[\#\text{ of jumps in } [0,t]\right] \leq C(p,\mathfrak{m},t)\gamma^p\epsilon^{-2}\alpha^{-1}\ .
\end{aligned}$$

And the proof is complete taking p large enough. To go from the first line to the second we used the fact that the rate of the Poisson processes controlling the jumps is a constant, hence it is not dependent from the process. □

Proof of Lemma 2.4.3. We will apply the Burkholder-Davis-Gundy inequality to $U^{(i,i)}_{\gamma,t}(s,x)$, defined in (2.75). The bracket process of $s \mapsto R_{\gamma,t}(s,x)$ is given by

$$s \mapsto \sum_{r \le s} (\Delta_r R_{\gamma,t}(r,x))^2 \tag{2.85}$$

and the jumps of $U^{(i,i)}_{\gamma,t}(s,x)$ are given by the jumps of $R_{\gamma,t}(r,x)$. Using Lemma 2.4.10

$$\mathbb{E}\left[\sup_{0\le r\le t} |\Delta_r U^{(i,i)}_{\gamma,t}(s,x)|^p\right] \lesssim \mathbb{E}\left[\sup_{0\le r\le t} |\Delta_r R_{\gamma,t}(s,x)|^{2p}\right] \le C(t,\mathfrak{m},\kappa)\gamma^{2p(1-\kappa)} \ .$$

It remains to control the quadratic variation $\left\langle U^{(i,i)}_{\gamma,t}(\cdot,x)\right\rangle_s$. We can write it as

$$\sum_{z\in\Lambda_\varepsilon} \int_{r=0}^{s} \left(\epsilon^2\delta^{-1} P^\gamma_{t-r} K_\gamma(x-z)\right)^4 \\ \times d\left\langle (\sigma^{(i)}_z(\alpha^{-1}r) - \sigma^{(i)}_z(\alpha^{-1}r^-))^2 \mathscr{I}^{r,z} - \alpha^{-1} Q^{i,i}_{\mathfrak{m}}(r^-,z)\right\rangle_r \tag{2.86}$$

where $\mathscr{I}^{r,z}$ is the Poisson process of rate α^{-1}, that is responsible for the jumps.
The quantity in the angled brackets in (2.86), is bounded by

$$\alpha^{-1} C \int_S |\eta_z(\alpha^{-1}r) - \sigma_z(\alpha^{-1}r)|^4 p_{\mathfrak{m}}(z,r^-,\sigma)(d\eta) \le C(\mathfrak{m})\alpha^{-1}\left(1 + |\sigma_z(\alpha^{-1}r^-)|^4\right) \ .$$

Using the general Hölder inequality and Remark 2.2.17

$$\mathbb{E}[\prod_{j=1}^{p/2} 1 + |\sigma_{z_j}(\alpha^{-1}r_j)|^4] \le \prod_{j=1}^{p/2}\left(\mathbb{E}[1+|\sigma_{z_j}(\alpha^{-1}r_j)|^4]^{p/2}\right)^{2/p} \le C(\mathfrak{m},p)$$

we find

$$\mathbb{E}\left[\left\langle U^{(i,i)}_{\gamma,t}(\cdot,x)\right\rangle_s^{p/2}\right] \le C(\mathfrak{m},p)\left(\alpha^{-1}\epsilon^8\delta^{-4}\sum_{z\in\Lambda_\varepsilon}\int_{r=0}^{s}\left(P^\gamma_{t-r}K_\gamma(x-z)\right)^4 dr\right)^{p/2}$$

and the conclusion follows from the fact $|P^\gamma_{t-s}K_\gamma(x)| \le \epsilon^2\gamma^{-2}$ for $x \in \Lambda_\varepsilon$ and Lemma B.0.4. By scaling (2.20), $\alpha \sim \epsilon^2\gamma^{-2}$ and

$$\mathbb{E}\left\langle U^{(i,i)}_{\gamma,t}(\cdot,x)\right\rangle_s^{p/2} \lesssim \left(\gamma^2\sum_{z\in\Lambda_\varepsilon}\epsilon^2\int_{r=0}^{s}\left(P^\gamma_{t-r}K_\gamma(x-z)\right)^2 dr\right)^{p/2} \lesssim \gamma^{p(1-\kappa)}$$

where the constants depends on $\mathfrak{m}, p, \kappa$ and the proof is completed. □

The next lemma correspond to [MW17a, Lemma 5.2].

Lemma 2.4.11 *For $j = 1, \dots, m$, any $t \geq 0$ and $1 \leq p < \infty$ and γ small enough*

$$\mathbb{E}\left[\left|\sup_{x\in\Lambda_\varepsilon}\sum_{r\leq t}|\Delta_r R^{(j)}_{\gamma,t}(r,x)|^2\right|^p\right]^{1/p} \leq C(t,p)\log(\gamma^{-1}) \ . \tag{2.87}$$

We have the following proposition, which correspond to [MW17a, Proposition 5.3].

Proof of Proposition 2.4.2. The proof uses the generalized multidimensional Itô formula for processes with finite first variation, that can be found in [Pro90, Chapter II]. Let $X_t = (X_{1,t}, \dots, X_{n,t})$ a multidimensional process with finite first variation and let $[X_i, X_j]_t$ be its bracket process (find citation). Then

$$\begin{aligned} f(X_t) = f(X_0) &+ \sum_j \int_0^t \partial_j f(X_{s^-}) dX_{j,s} + \frac{1}{2}\sum_{i,j}\int_0^t \partial_i\partial_j f(X_{s^-}) d[X_i, X_j]_s \\ &+ \sum_{s\leq t}\left(\Delta f(X_s) - \sum_j \partial_j f(X_{s^-})\Delta X_{j,s} - \sum_{i,j}\frac{\partial_j\partial_i f(X_{s^-})}{2}\Delta X_{i,s}\Delta X_{j,s}\right) \ . \end{aligned} \tag{2.88}$$

The key step in the proof uses the Itô formula to prove (2.76) by induction. Indeed for $\mathbf{k} = (0, \dots, 0)$, the error (2.73) is zero and (2.76) is trivially true. Recall the definitions of the Hermite polynomials $H_\mathbf{k}$ and its derivatives in Section 2.2.2 Using the Itô formula on $H_\mathbf{k}(\underline{R}_s) = H_\mathbf{k}(R_{\gamma,t}(s,x), [R_{\gamma,t}(\cdot,x)]_s)$

$$\begin{aligned} H_\mathbf{k}(\underline{R}_s) = &\sum_{i,j=1}^m \int_{r=0}^s \partial_{T_{i,j}} H_\mathbf{k}(\underline{R}_{r^-}) d[R^{(i)}_{\gamma,t}(\cdot,x), R^{(j)}_{\gamma,t}(\cdot,x)]_r \\ &+ \sum_{j=1}^m \int_{r=0}^s \partial_{X_j} H_\mathbf{k}(\underline{R}_{r^-}) dR^{(j)}_{\gamma,t}(r,x) \\ &+ \frac{1}{2}\sum_{i,j=1}^m \int_{r=0}^s \partial_{X_j}\partial_{X_i} H_\mathbf{k}(\underline{R}_{r^-}) d[R^{(i)}_{\gamma,t}(\cdot,x), R^{(j)}_{\gamma,t}(\cdot,x)]_r + Err_\mathbf{k}(s,x) \end{aligned}$$

where $Err_{\mathbf{k}}(s,x)$ contains the jumps.

$$Err_{\mathbf{k}}(s,x) = \sum_{r\le s}\Bigg(\Delta H_{\mathbf{k}}(\underline{R}_r) - \sum_{j=1}^m \partial_{X_j} H_{\mathbf{k}}(\underline{R}_{r^-})\Delta_r R^{(j)}_{\gamma,t}(\cdot,x)$$
$$- \sum_{i,j=1}^m \partial_{T_{i,j}} H_{\mathbf{k}}(\underline{R}_{r^-})\Delta_r[R^{(i)}_{\gamma,t}(\cdot,x), R^{(j)}_{\gamma,t}(\cdot,x)]_r$$
$$-\frac{1}{2}\sum_{i,j=1}^m \partial_{X_j}\partial_{X_i} H_{\mathbf{k}}(\underline{R}_{r^-})\Delta_r R^{(j)}_{\gamma,t}(\cdot,x)\Delta_r R^{(i)}_{\gamma,t}(\cdot,x)\Bigg) \ .$$

Using the proprieties of Hermite polynomials $\left(\frac{1}{2}\partial_{X_j}\partial_{X_i} + \partial_{T_{i,j}}\right) H_{\mathbf{k}}(\bar{x},\bar{T}) = 0$ the above can be rewritten as

$$H_{\mathbf{k}}(\underline{R}_s) = \sum_{j=1}^m k_j \int_{r=0}^s H_{\mathbf{k}^{j-}}(\underline{R}_{r^-}) dR^{(j)}_{\gamma,t}(r,x) + Err_{\mathbf{k}}(s,x)$$

which has the same form as (2.59). Subtracting the quantity $R^{:\mathbf{k}:}_{\gamma,t}$ we thus obtain

$$E^{:\mathbf{k}:}_{\gamma,t}(s,x) = \sum_{j=1}^m k_j \int_{r=0}^s E^{:\mathbf{k}^{j-}:}_{\gamma,t}(r^-,x) dR^{(j)}_{\gamma,t}(r,x) + Err_{\mathbf{k}}(s,x) \ . \tag{2.89}$$

We will use the induction over $|\mathbf{k}|$ to prove (2.74). Clearly (2.74) holds for every $\mathbf{k}$ with $|\mathbf{k}| = 1$. Assume that (2.74) holds for every multiindex $\mathbf{0} \le \mathbf{a} < \mathbf{k}$. We shall show that the conclusion of the proposition also holds for $\mathbf{k}$.

The first step consists in applying the Burkholder-Davis-Gundy inequality to the integral in (2.89). The quadratic variation is given by

$$\left\langle \int_{r=0}^{\cdot} E^{:\mathbf{k}^{j-}:}_{\gamma,t}(r^-,x) dR^{(j)}_{\gamma,t}(r,x)\right\rangle_s \le C\int_0^s \left|E^{:\mathbf{k}^{j-}:}_{\gamma,t}(r^-,x)\right|^2 d\left\langle R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_r$$
$$\le C\sup_{0\le r\le t}\left|E^{:\mathbf{k}^{j-}:}_{\gamma,t}(r^-,x)\right|^2 \left\langle R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_s$$

and using the Cauchy-Schwarz inequality, the expectation of the $\frac{p}{2}$-th power of the quantity above is bounded by

$$\mathbb{E}\left[\sup_{0\le r\le t}\left|E^{:\mathbf{k}^{j-}:}_{\gamma,t}(r^-,x)\right|^{2p}\right]^{\frac{1}{2}} \mathbb{E}\left[\left\langle R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_s^p\right]^{1/2}$$
$$\le C(\kappa,t,p,\mathfrak{m})\gamma^{p(1-\kappa/2)}\mathbb{E}\left[\left\langle R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_s^p\right]^{1/2} \le C(\kappa,t,p,\mathfrak{m})\gamma^{p(1-\kappa/2)}\gamma^{-p\kappa/2} \ ,$$

where we used induction and (2.62). We bound the jump term in a similar way, using Lemma 2.4.10

$$\mathbb{E}\left[\sup_{0\le s\le t}|\Delta_s E^{:\mathbf{k}:}_{\gamma,t}(s,x)|^p\right]\le \mathbb{E}\left[\sup_{0\le r\le t}|E^{:\mathbf{k}:^{j-}}_{\gamma,t}(r,x)|^p|\Delta_r R^{(j)}_{\gamma,t}(\cdot,x)|^p\right]$$
$$\le \mathbb{E}\left[\sup_{0\le r\le t}|E^{:\mathbf{k}:^{j-}}_{\gamma,t}(r,x)|^{2p}\right]^{\frac{1}{2}}\mathbb{E}\left[\sup_{0\le r\le t}|\Delta_r R^{(j)}_{\gamma,t}(\cdot,x)|^{2p}\right]^{\frac{1}{2}}\le C(\kappa,t,p,\mathfrak{m})\gamma^{p(1-\kappa)}\ .$$

It remains to bound the error $Err_{\mathbf{k}}(s,x)$, that contains the errors from the application of the Itô formula for processes with jumps, Taylor expanding up to second order the Hermite polynomials. For $\bar{x}=(x_1,\dots,x_m), t=(t_{i,j})_{i,j=1}^m$

$$\Bigg|H_{\mathbf{k}}(\bar{x}+\bar{y},\bar{t}+\bar{r})-H_{\mathbf{k}}(\bar{x},\bar{t})$$
$$-\sum_{j=1}^n\partial_{X_j}H_{\mathbf{k}}(\bar{x},\bar{t})y_j-\frac{1}{2}\sum_{i,j=1}^n\partial_{X_j}\partial_{X_i}H_{\mathbf{k}}(\bar{x},\bar{t})y_jy_i-\sum_{i,j=1}^n\partial_{T_{i,j}}H_{\mathbf{k}}(\bar{x},\bar{t})r_{i,j}\Bigg|$$
$$\le C\Big(\sum_{\mathbf{a}:|\mathbf{a}|=|\mathbf{k}|-2}|\bar{x}|^{\mathbf{a}}+|\bar{t}|^{\mathbf{a}}+1\Big)\Big(\sum_{\mathbf{b}:|\mathbf{b}|=3}|\bar{y}|^{\mathbf{b}}+\sum_{\mathbf{b}:|\mathbf{b}|=2}|\bar{r}|^{\mathbf{b}}\Big)$$

hence

$$|Err_{\mathbf{k}}(s,x)|\le C\sum_{j=1}^m\left(\sup_{r\le s}|R^{(j)}_{\gamma,t}(r,x)|^{|\mathbf{k}|-2}+\sup_{r\le s}[R^{(j)}_{\gamma,t}(\cdot,x)]_r^{(|\mathbf{k}|-2)/2}+1\right)$$
$$\times\sum_{r\le s}\left(|\Delta_r R^{(j)}_{\gamma,t}(r,x)|^3+|\Delta_r[R^{(j)}_{\gamma,t}(\cdot,x)]_r|^2\right)$$

and using Lemma 2.4.10 and Lemma 2.4.11, and Hölder inequality for $q_1^{-1}+q_2^{-1}+q_3^{-1}=p^{-1}$

$$\mathbb{E}\left[\sup_{x\in\Lambda_\varepsilon,s\in[0,t]}|Err_{\mathbf{k}}(s,x)|^p\right]^{1/p}\le\mathbb{E}\left[\left(\sup_{x\in\Lambda_\varepsilon,r\le s}|R_{\gamma,t}(r,x)|_{\mathbb{R}^n}^{|\mathbf{k}|-2}+1\right)^{q_1}\right]^{1/q_1}$$
$$\times\mathbb{E}\left[|\Delta_r R_{\gamma,t}(r,x)|^{q_2}\right]^{1/q_2}\mathbb{E}\left[\sup_{x\in\Lambda_\varepsilon}\left(\sum_{r\le s}|\Delta_r R_{\gamma,t}(r,x)|_{\mathbb{R}}^2\right)^{q_3}\right]^{1/q_3}$$
$$\le C(\kappa,\mathfrak{m},t,p)\gamma^{1-\kappa}E\left[\sup_{x\in\Lambda_\varepsilon,r\le s}|R_{\gamma,t}(r,x)|_{\mathbb{R}^n}^{q_1(|\mathbf{k}|-2)}+1\right]^{1/q_1}$$
$$\le C(\kappa,\mathfrak{m},t,p)\gamma^{1-\kappa}\gamma^{-\kappa}$$

where in the last line we used Remark 2.3.2 and the induction is proven. $\square$

Proof of Proposition 2.4.4. We will prove the above theorem for p large, the theorem for all $p > 1$ will follows from the monotonicity of L^p norms. We start computing

$$\begin{aligned}
&\left\langle R^{(i)}_{\gamma,t}(\cdot,x), R^{(j)}_{\gamma,t}(\cdot,x)\right\rangle_s - \mathfrak{c}_{\gamma,t}(s)\delta_{i,j} \\
&= \int_0^s \sum_{z_1,z_2\in\Lambda_\varepsilon} \epsilon^4 P^\gamma_{t-r}(x-z_1)P^\gamma_{t-r}(x-z_2)\left(d\left\langle M^{(i)}_\gamma(\cdot,z_1), M^{(j)}_\gamma(\cdot,z_2)\right\rangle_r - 2\delta_{i,j}dr\right) \\
&\qquad\qquad = \int_0^s \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |P^\gamma_{t-r}K_\gamma(x-z)|^2 \left(Q^{i,j}_{\mathfrak{m}}(r^-,z) - 2\delta_{i,j}\right) dr\,. \quad (2.90)
\end{aligned}$$

The proof consists in evaluating the difference between $Q^{i,j}_{\mathfrak{m}}(r^-,z)$ and $2\delta_{i,j}$. In order to prove that the average is negligible in the limit we will need to exploit the time integral. From Proposition 2.2.17, and the form of the stopping time $\tau_{\gamma,\mathfrak{m}}$ follows that we can prove the statement of Proposition 2.4.4 for

$$\int_0^s \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |P^\gamma_{t-r}K_\gamma(x-z)|^2 \left(\sigma^{(i)}_{\epsilon^{-1}z}(\alpha^{-1}s)\sigma^{(j)}_{\epsilon^{-1}z}(\alpha^{-1}s) - \delta_{i,j}\right) dr\,. \quad (2.91)$$

Following [SW16], we produce a coupling with the dynamic at infinite temperature $\beta = 0$. Let

$$Z_h = \int_S e^{\beta\langle h,\eta\rangle}\nu_\gamma(d\eta) \qquad P^h = \int_S Z_h^{-1}e^{\beta\langle h,\eta\rangle}\wedge 1\,\nu_\gamma(d\eta)$$

In particular

$$0 \le 1 - P^h \le 2\beta|h|\int_S |\eta|e^{\beta|h||\eta|}\nu_\gamma(d\eta)\,.$$

Let $\tilde\sigma_x(t)$ be a process on $S^{\Lambda_N}\times\mathbb{R}^+$, starting from the configuration with all spins equal to 0 following the Glauber dynamic with parameter $\beta = 0$. Recall the construction in Section 2.2 together with stopping time in Subsection 2.2.5. We will now define the coupling between $\tilde\sigma_x(t)$ and $\sigma_x(t)$ as follows: since the Poisson times between each jumps have been chosen to be independent of the configuration and with constant mean, we can construct $\tilde\sigma_x(t)$ in such a way that it has jumps at the same time and at the same place as the original process. Assume a jump happens at (x,t). If $t > \tau_{\gamma,\mathfrak{m}}$, we chose $\sigma_x(t) = \tilde\sigma_x(t)$ since both are chosen according to ν_γ. If $t \le \tau_{\gamma,\mathfrak{m}}$ with probability $P^{h_\gamma(x,\sigma_{t^-})}$ we choose $\sigma_x(t) = \tilde\sigma_x(t)$ distributed according to the density (here $h_\gamma(x,t) = h_\gamma(x,\sigma_{t^-})$)

$$(P^{h_\gamma(x,t)})^{-1}\left[Z^{-1}_{h_\gamma(x,t)}e^{\beta\langle h_\gamma(x,t),\eta\rangle}\wedge 1\right]\nu_\gamma(d\eta)$$

and with probability $1 - P^{h_\gamma(x,t)}$ we will draw $\tilde\sigma_x(t)$ and $\sigma_x(t)$ independently with density, respectively proportional to

$$\left[1 - Z_{h_\gamma(x,t)}^{-1} e^{\beta\langle h_\gamma(x,t),\eta\rangle}\right]^+ \nu_\gamma(d\eta) \qquad \text{and} \qquad \left[Z_{h_\gamma(x,t)}^{-1} e^{\beta\langle h_\gamma(x,t),\eta\rangle} - 1\right]^+ \nu_\gamma(d\eta)$$

Thus for any function $f : S \to \mathbb{R}$, for $x \in \Lambda_N, t \in \mathbb{R}^+$ and $p \geq 1$

$$|f(\sigma_x(t)) - f(\tilde{\sigma}_x(t))| \leq C \|f\|_\infty \gamma^{1-\nu} + 1_{\{t \leq T_0\}} |f(\sigma_x(0))|$$

where T_0 denotes the time of the first jump. For the inequality we used the fact that $\|h_\gamma(\cdot,t)\|_{L^\infty} \leq \gamma^{1-\nu}\mathfrak{m}$ for $t \leq \tau_{\gamma,\mathfrak{m}}$ and the fact that ν_γ has exponential moments.

$$\sum_{x\in\Lambda_\varepsilon} \mathbb{E}\left(\int_0^s \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |P_{t-r}^\gamma K_\gamma(x-z)|^2 \left|f(\sigma_{\epsilon^{-1}z}(\alpha^{-1}r)) - f(\tilde{\sigma}_{\epsilon^{-1}z}(\alpha^{-1}r))\right| dr\right)^p$$
$$\lesssim \epsilon^{-2}\left(\gamma^{1-\nu} + \mathbb{E}\left[\alpha T_0 t^{-1} \wedge 1\right]\right)^p \log^p(\gamma^{-1}) \ .$$

The last expectation is estimated with $x \wedge 1 \leq x^b$ for any $b \in [0,1]$. This implies that it is sufficient to prove the proposition in the infinite temperature case, starting from the zero initial condition. Let $\tau_l(x)_{x\in\Lambda_N, l\in\mathbb{N}}$ denote the collection of random times where $\tau_l(z)$ is the time at which the spin at site x jumps for the l-th time, in macroscopic coordinate. When a jump occurs, the distribution of the new spin is drawn independently from the other, according to ν_γ. Let M_s the quantity in (2.91), calculated with the auxiliary process $\tilde{\sigma}$. We bound the supremum of (2.78) in time with the supremum over a discretization of $[0,T]$ of mesh size γ^R where R is chosen later. The difference $|M_s - M_{\gamma^R\lfloor\gamma^{-R}s\rfloor}|$ is bounded by

$$2\int_{\gamma^R\lfloor\gamma^{-R}s\rfloor}^s \left\|P_{t-r}^\gamma K_\gamma\right\|_{L^2(\Lambda_N)}^2 dr \, \|\tilde{\sigma}\|_{L^\infty(\Lambda_N\times[0,T])}^2 \leq \gamma^R \epsilon^{-2}\gamma^2 \|\tilde{\sigma}\|_{L^\infty(\Lambda_N\times[0,T])}^2 \ .$$

Using

$$\mathbb{E}\Big[\sup_{z\in\Lambda_N;\, s\in[0,T])} |\tilde{\sigma}_z(\alpha^{-1}s)|^{2p}\Big] \lesssim \mathbb{E}\left[\text{\# of jumps in [0,T]}\right] = \epsilon^{-2}\alpha^{-1}T$$

we deduce that R has to satisfy $\gamma^R\epsilon^{-2}\gamma^2 \lesssim \gamma$. Bounding $\mathbb{E}[\sup_{s\in\gamma^R\mathbb{Z}\cap[0,t]} |M_s|^p]$ with $\mathbb{E}[\sum_{s\in\gamma^R\mathbb{Z}\cap[0,t]} |M_s|^p]$ it remains to estimate $\mathbb{E}[|M_s|^p]$. Let us expand the product

$$\sum_{z_1,\ldots,z_p\in\Lambda_N} \epsilon^{2p} \sum_{l_1,\ldots,l_p\geq 1} \mathbb{E}\prod_{v=1}^p \left(\int_{\tau_{l_v}(z_v)\wedge s}^{\tau_{l_v+1}(z_v)\wedge s} |P_{t-r}^\gamma K_\gamma(x-z)|^2 [\tilde{\sigma}^{(i)}\tilde{\sigma}_{z_v}^{(j)}(\alpha^{-1}\tau_{l_v}) - \delta_{i,j}] dr\right)$$

and notice that for different z_v or l_v, the quantity inside the integrals are independent and with mean zero. We can thus perform the summation indexed over the possible partition of $\{1,\dots,p\}$ that don't contain singletons. Let p be even and denote with $\mathcal{P}^*$ the set of such partitions, let $(q_1,\dots,q_m)$ be the sizes of the sets of a given partition $Q\in\mathcal{P}^*$ with $q_1+\dots+q_m=p$

$$\begin{aligned}
&\epsilon^{2(p-m)}\sum_{z_1,\dots,z_m\in\Lambda_N}\epsilon^{2m}\sum_{l_1,\dots,l_m\geq 1}\prod_{v=1}^m\mathbb{E}\left[\left(\int_{\tau_{l_v}(z_v)\wedge s}^{\tau_{l_v+1}(z_v)\wedge s}|P^\gamma_{t-r}K_\gamma(x-z)|^2dr\right)^{q_v}\right]\\
&\leq\epsilon^{2(p-m)}\left(\int_0^s\left\|P^\gamma_{t-r}K_\gamma\right\|^2_{L^2(\Lambda_\varepsilon)}dr\right)^m\left\|P^\gamma_{t-r}K_\gamma\right\|^{2(p-m)}_{L^\infty(\Lambda_\varepsilon)}\mathbb{E}\Big[\sup_{z,l}|\tau_l(z)-\tau_{l+1}(z)|^{p-m}\Big]\\
&\qquad\lesssim\epsilon^{2(p-m)}\log^m(\gamma^{-1})(\epsilon^{-2}\gamma^2)^{2(p-m)}(\epsilon^{-2}\alpha^{-1}T)\alpha^{p-m}\\
&\qquad\qquad\lesssim\log^m(\gamma^{-1})\gamma^{2(p-m)}\epsilon^{-2}\alpha^{-1}
\end{aligned}$$

where the supremum runs over $z\in\Lambda_N$ and $l\geq 1$ such that $\tau_l(z)\leq s$. Here in the second inequality we used lemma B.0.4. Choosing p large enough $m\leq p/2$ proves the proposition.

□

Proof of Proposition 2.4.7. In virtue of Lemma 2.4.3 and bound (2.74), it is sufficient to show the above inequality for the difference

$$H_{\mathbf{k}}(R_{\gamma,t}(s,z),\mathfrak{c}_{\gamma,t}(s)I_m)-H_{\mathbf{k}}(R_{\gamma,t}(s,z),[R_{\gamma,t}(\cdot,z),R_{\gamma,t}(\cdot,z)]_s)\ .$$

It is easy to see that the above difference can be written as a polynomial in the entries of the matrix $\mathfrak{c}_{\gamma,t}(s)I_m-[R_{\gamma,t}(\cdot,z),R_{\gamma,t}(\cdot,z)]_s$. The coefficient of the polynomial are of the form $\partial_{T_{i_1,i_2}}\partial_{T_{i_{m-1},i_m}}H_{\mathbf{k}}(R_{\gamma,t}(s,z),[R_{\gamma,t}(\cdot,z),R_{\gamma,t}(\cdot,z)]_s)$.
Using the recursion formula for the Hermite polynomials in Subsection 2.2.2, we can bound the left-hand-side of (2.81) with

$$\begin{aligned}
&\leq C(\mathbf{k},p)\sup_{\substack{\mathbf{0}\leq\mathbf{a}\leq\mathbf{k}\\1\leq i,j\leq m}}\mathbb{E}\left[\sup_{x\in\Lambda_\varepsilon}\sup_{0\leq s\leq t}\left|[R^{(i)}_{\gamma,t}(\cdot,x),R^{(j)}_{\gamma,t}(\cdot,x)]_s-\mathfrak{c}_{\gamma,t}(s)\delta_{i,j}\right|^{2p\left\lfloor\frac{|\mathbf{k}-\mathbf{a}|}{2}\right\rfloor}\right]^{1/2p}\\
&\qquad\times\mathbb{E}\left[\sup_{x\in\Lambda_\varepsilon}\sup_{0\leq s\leq t}|H_{\mathbf{a}}(R_{\gamma,t}(s,z),[R_{\gamma,t}(\cdot,z),R_{\gamma,t}(\cdot,z)]_s)|^{2p}\right]^{1/2p}\\
&\leq C(\kappa,T,p,\mathfrak{m},\nu,\mathbf{k},b)\gamma^{-\kappa}\left(\gamma^{(1-\nu-\kappa)}+(1\wedge t^{-b}\alpha^b)\left(\mathbb{E}\sup_x|\sigma_{\epsilon^{-1}x}(0)|^{2p|\mathbf{k}|}\right)^{1/2p}\right)
\end{aligned}$$

where in the last line we used Propositions 2.4.4, 2.4.2 and 2.3.4 with the observation B.0.5. The proof then follows from assumption (I2') and (2.38) for a suitably large power.

□

2.5 The nonlinear process

In this section we prove Theorem 2.2.21 controlling the nonlinear dynamic. In Section 2.4, we showed that the process Z_γ, obtained from the dynamic stopped at random time $\tau_{\gamma,\mathfrak{m}}$, as described in Subsection 2.2.4, is convergent in law to the vector-valued stochastic heat equation. The random time guarantees a control over the $\mathcal{C}^{-\nu}$ norm of the process for a given $\nu > 0$, that we will assume to be fixed for this section. Following the strategy outlined in Subsection 2.2.6, we use the linear dynamic to control the nonlinear one.
Recall from (2.17) in Section 2.2 that the nonlinear process X_γ, started from X^0_γ satisfies

$$X_\gamma(z,t) = P^\gamma_t X^0_\gamma(z) + \int_0^t P^\gamma_{t-s} K_\gamma *_\epsilon \left(\tilde{\mathfrak{p}}_\gamma(X_\gamma(z,s)) + E_\gamma(z,s)\right) ds + Z_\gamma(z,t) \quad (2.92)$$

for $z \in \Lambda_\varepsilon, t \in [0,T]$, and where the polynomial $\tilde{\mathfrak{p}}_\gamma$ is defined in (2.22).

2.5.1 Renormalization of the polynomial

At some point it will be more convenient to renormalize the power of X_γ with the time dependent $\mathfrak{c}_\gamma(s)$ approximation of $\mathfrak{c}_\gamma$ defined in (2.80). Consequently we will define

$$\mathfrak{a}^\gamma_{2k+1}(s) \stackrel{\text{def}}{=} \left(e^{\frac{\mathfrak{c}_\gamma(s)}{2}\Delta^*_X}\tilde{\mathfrak{a}}^\gamma\right)_{2k+1} \quad (2.93)$$

and the corresponding decomposition

$$\begin{aligned}\mathfrak{p}^{(j)}_\gamma(X_\gamma(z,s),s) \stackrel{\text{def}}{=} \mathfrak{a}^\gamma_1(s)X^{(j)}_\gamma(z,s) + \mathfrak{a}^\gamma_3(s)H(X^{(j)}_\gamma|X_\gamma|^2,\mathfrak{c}_\gamma(s))(z,s) + \dots \\ \dots + \mathfrak{a}^\gamma_{2n-1}(s)H(X^{(j)}_\gamma|X_\gamma|^{2n-2},\mathfrak{c}_\gamma(s))(z,s) \quad (2.94)\end{aligned}$$

The two similar decompositions (2.34) and (2.94) will be useful for different purposes, in particular (2.94) will be used when we will separate the linear part of the dynamic from the nonlinear one.
We now provide a description for the aforementioned polynomials as γ goes to zero.
Assumption (M1) guarantees that the limit of $\mathfrak{a}^\gamma_{2k+1}$ is well defined. Moreover, from (2.94) and (M1) we have that the following limit exists for every $s > 0$

$$\mathfrak{a}_{2k+1}(s) \stackrel{\text{def}}{=} \lim_{\gamma\to 0}\mathfrak{a}^\gamma_{2k+1}(s) = \lim_{\gamma\to 0}\left(e^{\frac{\mathfrak{c}_\gamma(s)-\mathfrak{c}_\gamma}{2}\Delta^*_X}\mathfrak{a}^\gamma\right)_{2k+1} = \left(e^{\frac{A(s)}{2}\Delta^*_X}\mathfrak{a}\right)_{2k+1}$$

and

$$
\begin{aligned}
&|\mathfrak{a}_{2k+1}(s) - \mathfrak{a}^{\gamma}_{2k+1}(s)| \\
&\quad \leq \left| \left(\left(e^{\frac{A(s)}{2}\Delta^*_X} - e^{\frac{\mathfrak{c}_\gamma(s)-\mathfrak{c}_\gamma}{2}\Delta^*_X} \right) \mathfrak{a}^{\gamma} \right)_{2k+1} \right| + \left| \left(e^{\frac{A(s)}{2}\Delta^*_X} \left(\mathfrak{a} - \mathfrak{a}^{\gamma} \right) \right)_{2k+1} \right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lesssim s^{-\lambda}\alpha^{\lambda} |A(s)|^{|\mathbf{k}|-1} + |A(s)|^{|\mathbf{k}|} c_0 \gamma^{\lambda_0}
\end{aligned}
$$

where $A(s) \stackrel{\text{def}}{=} \lim_{\gamma\to 0} \mathfrak{c}_\gamma(s) - \mathfrak{c}_\gamma = \frac{s}{2} - \sum_{0<|\omega|} \frac{e^{-2\pi^2 s|\omega|^2}}{4\pi^2|\omega|^2}$ is a continuous function in s on $(0, T]$ that diverges logarithmically as $s \to 0$. Here we used the bounds in [MW17a, Lemma 7.1]

$$
|A(s) - \mathfrak{c}_\gamma(s) + \mathfrak{c}_\gamma| \lesssim s^{-\lambda}\alpha^{\lambda}
$$

for $\lambda \in (0, 1/2)$.

2.5.2 Approximation and convergence of the nonlinear dynamic

We will now introduce some approximations of the nonlinear part of the process.
We will first extend to the whole torus the relation (2.92), in the same way as in [MW17a]. This is not automatic since taking the power of the field do not commute with the trigonometric polynomial extension. In doing so recall the extension operator defined in (1.9) and the definitions (2.54). Consider moreover the convolution

$$
F \star G(z) = \int_{[-1,1]^2} F(x-y)G(y)dy
$$

for $x \in [-1, 1]^2$.

Proposition 2.5.1 *The multidimensional process X_γ, extended over the torus as in* (1.9)*, started from X^0_γ satisfies*

$$
X_\gamma(z,t) = P^\gamma_t X^0_\gamma(z) + \int_0^t P^\gamma_{t-s} K_\gamma \star \left(\tilde{\mathfrak{p}}_\gamma(X_\gamma(\cdot, s)) + \mathsf{Err}(\cdot, s) \right)(z) ds + Z_\gamma(z,t) \tag{2.95}
$$

for $z \in \mathbb{T}^2, t \in [0,T]$, where the polynomial $\tilde{\mathfrak{p}}_\gamma$ is defined in (2.22)*.*
Moreover the error term satisfies

$$
\begin{aligned}
&\|\mathsf{Err}(\cdot, s)\|_{L^\infty(\mathbb{T}^2)} \leq C(T,\nu,\kappa) \left(1 + \|X_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}}\right)^{2n-2} \times \\
&\qquad\qquad \times \left(\gamma^{-\kappa} \epsilon^{-2(n-1)\nu} \left\| X^{high}_\gamma(\cdot, s) \right\|_{L^\infty(\mathbb{T}^2)} + \gamma^2 \epsilon^{-(2n+1)\nu} \|X_\gamma(\cdot,s)\|^3_{\mathcal{C}^{-\nu}} \right)
\end{aligned}
$$

Proof. The proof is the same as [MW17a, Lemma 7.1], we only recall the bound on the error. From (2.23) and Lemma B.0.5, for $x \in \Lambda_\varepsilon$

$$|E_\gamma(x,s)| \leq C(\mathfrak{m},\nu)\gamma^2\epsilon^{-(2n+1)\nu}\,\|X_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}}^{2n+1}$$

and we can extend the previous inequality to $x \in \mathbb{T}^2$ at the expenses of an arbitrary small negative power of ϵ. □

Corollary 2.5.2 *Let $c_0 > 0$ and $\lambda_0 > 0$ as in* (M1)*. Then the process X_γ satisfies* (2.95) *with the limiting polynomial $\mathfrak{p}$ (whose coefficient are independent of γ) defined in* (2.25) *and the error term satisfies*

$$\begin{aligned}\|\mathsf{Err}(\cdot,s)\|_{L^\infty(\mathbb{T}^2)} \leq C(T,\nu,\kappa,\mathfrak{m})\left(1+\|X_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}}\right)^{2n-2}\\ \times\Big(\gamma^{-\kappa}\epsilon^{-2(n-1)\nu}\left\|X_\gamma^{high}(\cdot,s)\right\|_{L^\infty(\mathbb{T}^2)}+\dots\\ \dots+\gamma^2\epsilon^{-(2n+1)\nu}\|X_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}}^3+c_0\gamma^{\lambda_0-\kappa}\|X_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}}\Big)\end{aligned}$$

2.5.3 Da Prato - Debussche trick

We are now ready to apply the idea of Da Prato and Debussche [DPD03] in our context, as it was applied in [MW17a]. As described in Subsection 2.2.6, the trick relies in the decomposition of the solution X_γ into the linear term Z_γ approximation of the stochastic heat equation, and a remainder with finite quadratic variation, solving a PDE problem with random coefficients.

The treatment follows closely [MW17a] and [SW16], the only difference is given by the fact that in our case the process is multidimensional and an arbitrary quantity of Wick powers have to be controlled.

For $0 \leq t \leq T$ we will define the following approximation

$$\bar{X}_\gamma(\cdot,t) \stackrel{\text{def}}{=} P_tX^0(\cdot)+Z_\gamma(\cdot,t)+\mathcal{S}_T\left(\left(Z_\gamma^{:\mathbf{k}:}\right)_{|\mathbf{k}|\leq 2n-1}\right)(\cdot,t) \tag{2.96}$$

where X^0 is the initial condition for the continuous process (see also Assumption I1) and $\mathcal{S}_T$ is the solution map described in Subsection 2.2.6. Recall that, for any $\kappa > 0$ and $\nu > 0$, $\mathcal{S}_T$ is Lipschitz continuous from $L^\infty([0,T];(\mathcal{C}^{-\nu})^{n^*})$ to $\mathcal{C}([0,T];\mathcal{C}^{2-\nu-\kappa})$ with $n^* = \binom{2n-2+m}{m-1}$. In particular, by theorem 2.82 we have that the process $\bar{X}_\gamma$ converges in distribution to the solution X of the SPDE (2.46) as described in theorem 2.2.20.

Since

$$\left\|P_tX^0 - P_t^\gamma X_\gamma^0\right\|_{\mathcal{C}^{-\nu}} \leq \left\|P_t(X^0-X_\gamma^0)\right\|_{\mathcal{C}^{-\nu}} + \left\|(P_t-P_t^\gamma)X_\gamma^0\right\|_{\mathcal{C}^{-\nu}}\ .$$

From [MW17a, Lemma 7.3] we have that

$$\lim_{\gamma\to 0}\sup_{0\le t\le T}\left\|P_tX^0-P_t^\gamma X_\gamma^0\right\|_{\mathcal{C}^{-\nu}}=0\ .$$

It remains to control the difference between

$$v_\gamma(x,t)=X_\gamma(x,t)-Z_\gamma(x,t)-P_t^\gamma X_\gamma^0(x,t) \tag{2.97}$$

$$\bar v_\gamma(x,t)=\bar X_\gamma(x,t)-Z_\gamma(x,t)-P_tX^0(x,t)\ . \tag{2.98}$$

For $t\in[0,T]$, $x\in\mathbb{T}^2$. In order to do so, it is more convenient to start the remainder processes v_γ and $\bar v_\gamma$ from zero and add the initial condition to the martingales. This can be done rearranging the contribution of the initial condition and defining, for $\mathbf{k}\in\mathbb{N}^m$

$$\tilde Z_\gamma \stackrel{\text{def}}{=} P_t^\gamma X_\gamma^0+Z_\gamma \qquad\qquad \bar Z_\gamma^{:\mathbf{k}:}\stackrel{\text{def}}{=}\sum_{\substack{\mathbf{a}\in\mathbb{N}^m\\ \mathbf{a}\le\mathbf{k}}}(P_tX^0)^{\mathbf{a}}Z_\gamma^{:\mathbf{k}-\mathbf{a}:}\ . \tag{2.99}$$

The last relation is similar to (2.30) for the Hermite polynomial and

$$H\left((\tilde Z_\gamma+v_\gamma)^{\mathbf{k}},\mathfrak{c}_\gamma\right)=\sum_{\substack{\mathbf{a}\in\mathbb{N}^m\\ \mathbf{a}\le\mathbf{k}}}v_\gamma^{\mathbf{a}}H\left(\tilde Z_\gamma^{\mathbf{k}-\mathbf{a}},\mathfrak{c}_\gamma\right)\ .$$

Recall the heat kernel regularization proprieties of [MW17a, Cor. 8.7], for $\lambda>-\nu$

$$\left\|(P_tX^0)^{(j)}\right\|_{\mathcal{C}^\lambda}\le C(\lambda)t^{-\frac{\lambda+\nu}{2}}\left\|X^{0(j)}\right\|_{\mathcal{C}^{-\nu}}\le C(\lambda)t^{-\frac{\lambda+\nu}{2}}\left\|X^0\right\|_{(\mathcal{C}^{-\nu})^m}$$

and the Besov multiplicative inequality

$$\begin{aligned}\left\|\bar Z_\gamma^{:\mathbf{k}:}(\cdot,t)\right\|_{\mathcal{C}^{-\nu}}&\le C(\nu,\kappa)\sum_{\substack{\mathbf{a}\in\mathbb{N}^m\\ \mathbf{a}\le\mathbf{k}}}\sup_{j=1,\dots,m}\left\|(P_tX^0)^{(j)}\right\|_{\mathcal{C}^{\nu+\kappa}}^{|\mathbf{a}|}\left\|Z_\gamma^{:\mathbf{k}-\mathbf{a}:}(\cdot,t)\right\|_{\mathcal{C}^{-\nu}}\\&\le C(\nu,\kappa)\sum_{\substack{\mathbf{a}\in\mathbb{N}^m\\ \mathbf{a}\le\mathbf{k}}}t^{-\left(\nu+\frac{\kappa}{2}\right)|\mathbf{a}|}\left\|X^0\right\|_{(\mathcal{C}^{-\nu})^m}^{|\mathbf{a}|}\left\|Z_\gamma^{:\mathbf{k}-\mathbf{a}:}(\cdot,t)\right\|_{\mathcal{C}^{-\nu}}\\&\le C\left(n,\nu,\kappa,T,\left\|X^0\right\|_{(\mathcal{C}^{-\nu})^m}\right)t^{-\left(\nu+\frac{\kappa}{2}\right)(2n-1)}\sup_{\substack{\mathbf{a}\in\mathbb{N}^m\\ |\mathbf{a}|\le 2n-1}}\left\|Z_\gamma^{:\mathbf{a}:}(\cdot,t)\right\|_{\mathcal{C}^{-\nu}}\end{aligned} \tag{2.100}$$

for every $\mathbf{k}\in\mathbb{N}^m$ with $|\mathbf{k}|\le 2n-1$, the degree of the polynomial $\mathfrak{p}$.
This allows us to write, using the definition of $\mathcal{S}_T$ in Section 2.2.6, the relation for the j-th

component of (2.97) and (2.98)

$$\bar{v}_{\gamma}^{(j)}(\cdot,t) = \int_0^t P_{t-s}\overline{\Psi}^{(j)}\left(s,(\bar{Z}_{\gamma}^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(\bar{v}_{\gamma}(\cdot,s))\,ds$$

while a similar expression holds for $v_{\gamma}^{(j)}$ in virtue of Proposition 2.5.1 and corollary 2.5.2

$$v_{\gamma}^{(j)}(\cdot,t) = \int_0^t P_{t-s}^{\gamma}K_{\gamma}\star\left(\mathfrak{p}^{(j)}(\tilde{Z}_{\gamma}(\cdot,s)+v_{\gamma}(\cdot,s))+\mathsf{Err}_1^{(j)}(\cdot,s)\right)ds$$

with Err_1 satisfying the bound in corollary 2.5.2. The next proposition allows a control over the nonlinear part in a space of functions rather than distributions. It corresponds to in [MW17a, Lemma 7.5] and [SW16, Lemma 4.8]

Proposition 2.5.3 *There exists sufficiently small $\nu>0$ and $\kappa>0$, such that for all $T>0$ the following inequality holds*

$$\begin{aligned}\left\|\bar{v}_{\gamma}^{(j)}(\cdot,t)-v_{\gamma}^{(j)}(\cdot,t)\right\|_{\mathcal{C}^{\frac12}} \le C_1\int_0^t (t-s)^{-\frac13}s^{-\frac16}\left\|\bar{v}_{\gamma}(\cdot,s)-v_{\gamma}(\cdot,s)\right\|_{(\mathcal{C}^{1/2})^m}ds\\ + C_1\left(\left\|X^0-X_{\gamma}^0\right\|_{\mathcal{C}^{-\nu}}+\gamma^{\frac12}+\gamma^{\lambda_0-2\kappa}\right)+C_2|\overline{\mathsf{Err}_2}(t)| \quad (2.101)\end{aligned}$$

and the error term satisfies

$$\mathbb{E}\left[\sup_{t\le T}|\overline{\mathsf{Err}_2}(t)|\right] \le C(T,\nu,\mathfrak{m},\kappa,n)(\gamma\epsilon^{-\kappa}+\gamma^{1-\kappa}) \quad (2.102)$$

with the constant C_1 depending on $\nu,\kappa,T,n,\left\|X_{\gamma}^0\right\|_{(\mathcal{C}^{-\nu})^m}, \sup_{s\le T}\left\|Z_{\gamma}^{:\mathbf{a}:}(\cdot,s)\right\|_{(\mathcal{C}^{-\nu})^m}$ with $|\mathbf{a}|\le 2n-1$ and $\sup_{s\le T}\|v_{\gamma}(\cdot,s)\|_{(\mathcal{C}^{1/2})^m}$, while C_2 depends on T,ν,κ,n.

Proof. We are going to give a complete proof of this bound since it is the central ingredient for the proof of the main theorem. To keep the formulas light, we will use L^p in place of $L^p(\mathbb{T}^2)$. Decompose the difference into

$$\begin{aligned}\bar{v}_{\gamma}^{(j)}(\cdot,t)-v_{\gamma}^{(j)}(\cdot,t) = \int_0^t\left(P_{t-s}-P_{t-s}^{\gamma}K_{\gamma}\right)\star\overline{\Psi}^{(j)}\left(s,(\bar{Z}_{\gamma}^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(\bar{v}_{\gamma}(\cdot,s))\,ds\\ +\int_0^t P_{t-s}^{\gamma}K_{\gamma}\star\left(\overline{\Psi}^{(j)}\left(s,(\bar{Z}_{\gamma}^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(\bar{v}_{\gamma}(\cdot,s))-\mathfrak{p}^{(j)}\left(\tilde{Z}_{\gamma}(\cdot,s)+v_{\gamma}(\cdot,s)\right)\right)ds\\ -\int_0^t P_{t-s}^{\gamma}K_{\gamma}\star\mathsf{Err}_2^{(j)}(\cdot,s)ds\,. \quad (2.103)\end{aligned}$$

The first term in (2.103) is bounded in $\mathcal{C}^{\frac12}$ using Lemma B.0.8 with λ and κ satisfying

$-\lambda-\frac{1}{4}-\frac{\nu}{2}-\kappa>-1$ and $\left(\nu+\frac{\kappa}{2}\right)(2n-1)<1$

$$\begin{aligned}
&\int_0^t \left\|\left(P_{t-s}-P^\gamma_{t-s}K_\gamma\right)\star\overline{\Psi}^{(j)}\left(s,(\bar{Z}_\gamma^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(\bar{v}_\gamma(\cdot,s))\right\|_{\mathcal{C}^{\frac{1}{2}}}ds\\
&\le C(T,\lambda,\kappa)\int_0^t (t-s)^{-\lambda-\frac{1}{4}-\frac{\nu}{2}-\kappa}\gamma^\lambda\left\|\overline{\Psi}^{(j)}\left(s,(\bar{Z}_\gamma^{:\mathbf{k}:})_{|\mathbf{k}|\le 2n-1}\right)(\bar{v}_\gamma(\cdot,s))\right\|_{L^\infty([0,T],\mathcal{C}^{-\nu})}ds\\
&\qquad\le C\gamma^\lambda\int_0^t (t-s)^{-\lambda-\frac{1}{4}-\frac{\nu}{2}-\kappa}s^{-\left(\nu+\frac{\kappa}{2}\right)(2n-1)}ds\le C\gamma^\lambda \qquad (2.104)
\end{aligned}$$

where the last constant depends on $T,\lambda,\nu,\kappa,\|\bar{v}_\gamma\|_{L^\infty([0,T];\mathcal{C}^{\frac{1}{2}})},\left\|Z_\gamma^{:\mathbf{a}:}\right\|_{L^\infty([0,T];\mathcal{C}^{-\nu})}$.
The third part of (2.103)

$$\sup_{0\le t\le T}\left\|\int_0^t P^\gamma_{t-s}K_\gamma\star\mathsf{Err}_1^{(j)}(\cdot,s)ds\right\|_{\mathcal{C}^{\frac{1}{2}}}\le C(T)\int_0^T (T-s)^{-\frac{1}{4}-\kappa}\left\|\mathsf{Err}_2^{(j)}(\cdot,s)\right\|_{L^\infty}ds$$

is bounded with corollary 2.5.2: in particular we will need the following bounds provided by Lemma B.0.8

$$\|X_\gamma(\cdot,s)\|_{(\mathcal{C}^{-\nu})^m}\le\|Z_\gamma(\cdot,s)\|_{(\mathcal{C}^{-\nu})^m}+\left\|P^\gamma_s X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m}+\|v_\gamma(\cdot,s)\|_{(\mathcal{C}^{1/2})^m}$$

$$\begin{aligned}
&\left\|X_\gamma^{high}(\cdot,s)\right\|_{(L^\infty)^m}\le\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|_{(L^\infty)^m}+\left\|(P^\gamma_s X^0_\gamma)^{high}\right\|_{(L^\infty)^m}+\left\|v_\gamma^{high}(\cdot,s)\right\|_{(L^\infty)^m}\\
&\lesssim\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|_{(L^\infty)^m}+\left(\epsilon\gamma^{-1}\right)^\lambda t^{-\lambda-\frac{n\nu}{2(n-1)}}\left\|X^0_\gamma\right\|_{\mathcal{C}^{-\nu}}+\left(\epsilon\gamma^{-1}\right)^{\frac{1}{2}}\|v_\gamma(\cdot,s)\|_{(\mathcal{C}^{1/2})^m}
\end{aligned}$$

that implies that the left hand side of (2.102) is bounded by

$$\le C\left(\gamma^{-\kappa}\epsilon^{-2(n-1)\nu}\left(\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|_{(L^\infty)^m}+\left(\epsilon\gamma^{-1}\right)^{1/2}\right)+\gamma^2\epsilon^{-(2n+1)\nu}+\gamma^{\lambda_0-\kappa}\right)$$

for $\lambda=1/2$ and values of ν and κ small enough. The value of the constant C depends on $T,\nu,\kappa,n,\sup_{s\le T},\left\|X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m}$ and it is a polynomial function of the random quantities $\|Z_\gamma(\cdot,s)\|_{(\mathcal{C}^{-\nu})^m},\sup_{s\le T}\|v_\gamma(\cdot,s)\|_{(\mathcal{C}^{1/2})^m}$. Using the inequality $Cab\le\frac{1}{2}(C^2\gamma a^2+\gamma^{-1}b^2)$, we separate the constant and $\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|_{(L^\infty)^m}$. In particular we have that the error is controlled, changing the value of the constant if needed, by

$$\begin{aligned}
&\sup_{t\le T}\left\|\int_0^t P^\gamma_{t-s}K_\gamma\star\mathsf{Err}_2^{(j)}(\cdot,s)ds\right\|_{\mathcal{C}^{\frac{1}{2}}}\\
&\qquad\le C\gamma^{-\kappa'}\left(\gamma^{1\vee\lambda_0}+(\epsilon\gamma^{-1})^{1/2}\right)+\gamma^{-1}\sup_{s\le T}\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|^2_{(L^\infty)^m}ds \qquad (2.105)
\end{aligned}$$

for κ' small enough dependent on κ, ν, n. Lemma 2.3.7 offers a control in expectation of the high frequencies of Z_γ and completes the treatment of the error term.
It remains to bound the second term of (2.103). In order to do this we make use of the expression for $\overline{\Psi}$ defined in (2.51)

$$\overline{\Psi}^{(j)}\left(s, (\bar{Z}_\gamma^{:\mathbf{k}:})_{|\mathbf{k}|\leq 2n-1}\right)(\bar{v}_\gamma(\cdot,s)) = \sum_{|\mathbf{b}|+|\mathbf{a}|\leq 2n-1} b_{\mathbf{a},\mathbf{b}}^{(j)}(s)\bar{v}_\gamma^{\mathbf{a}}(\cdot,s)\bar{Z}_\gamma^{:\mathbf{b}:}(\cdot,s)$$

and using (2.99)

$$\begin{aligned}
&\overline{\Psi}^{(j)}\left(s, (\bar{Z}_\gamma^{:\mathbf{k}:})_{|\mathbf{k}|\leq 2n-1}\right)(\bar{v}_\gamma(\cdot,s)) - \mathfrak{p}^{(j)}\left(\tilde{Z}_\gamma(\cdot,s)+v_\gamma(\cdot,s)\right) \\
&\qquad = \sum_{|\mathbf{b}|+|\mathbf{a}|\leq 2n-1} b_{\mathbf{a},\mathbf{b}}^{(j)}(s)\left(\bar{v}_\gamma^{\mathbf{a}}(\cdot,s)\bar{Z}_\gamma^{:\mathbf{b}:}(\cdot,s) - v_\gamma^{\mathbf{a}}(\cdot,s)\tilde{Z}_\gamma^{:\mathbf{b}:}(\cdot,s)\right) \\
&= \sum_{\mathbf{a},\mathbf{b},\mathbf{c}} b_{\mathbf{a},\mathbf{b},\mathbf{c}}^{(j)}(s)\left(\bar{v}_\gamma^{\mathbf{a}}(\cdot,s)(P_sX^0)^{\mathbf{b}}(\cdot,s)Z_\gamma^{:\mathbf{c}:}(\cdot,s) - v_\gamma^{\mathbf{a}}(\cdot,s)(P_s^\gamma X_\gamma^0)^{\mathbf{b}}(\cdot,s)H_{\mathbf{c}}(Z_\gamma(s),\mathfrak{c}_\gamma(s))\right)
\end{aligned} \tag{2.106}$$

for some real $b_{\mathbf{a},\mathbf{b},\mathbf{c}}^{(j)}(s)$ growing like a power of $\log(s^{-1})$ as $s\to 0$, and satisfying $|b_{\mathbf{a},\mathbf{b},\mathbf{c}}^{(j)}(s)| \leq C(T,\kappa)s^{-\kappa}$ for $s\in[0,T]$. It is sufficient to bound each term in the sum of (2.106). Using the Besov multiplicative inequality A.0.5,

$$\begin{aligned}
&\left\|\left(\bar{v}_\gamma^{\mathbf{a}}(\cdot,s) - v_\gamma^{\mathbf{a}}(\cdot,s)\right)(P_sX^0)^{\mathbf{b}}(\cdot,s)Z_\gamma^{:\mathbf{c}:}(\cdot,s)\right\|_{\mathcal{C}^{-\nu}} \\
&\qquad\qquad \leq \left\|\bar{v}_\gamma^{\mathbf{a}}(\cdot,s) - v_\gamma^{\mathbf{a}}(\cdot,s)\right\|_{\mathcal{C}^{\frac{1}{2}}}\left\|(P_sX^0)^{\mathbf{b}}(\cdot,s)Z_\gamma^{:\mathbf{c}:}(\cdot,s)\right\|_{\mathcal{C}^{-\nu}} \\
&\lesssim \left\|\bar{v}_\gamma(s)-v_\gamma(s)\right\|_{(\mathcal{C}^{\frac{1}{2}})^m}\left\||\bar{v}_\gamma(s)|+|v_\gamma(s)|\right\|_{(\mathcal{C}^{\frac{1}{2}})^m}^{|\mathbf{a}|-1}\left\|P_sX^0(s)\right\|_{(\mathcal{C}^{\nu+\kappa})^m}^{|\mathbf{b}|}\left\|Z_\gamma^{:\mathbf{c}:}(s)\right\|_{\mathcal{C}^{-\nu}}
\end{aligned}$$

where $\left\|P_sX^0(s)\right\|_{(\mathcal{C}^{\nu+\kappa})^m} \leq C(\nu,\kappa)s^{-2\nu-\kappa}\left\|X^0\right\|_{(\mathcal{C}^{-\nu})^m}$. And similarly

$$\begin{aligned}
&\left\|v_\gamma^{\mathbf{a}}(s)\left((P_sX^0)^{\mathbf{b}}(s) - (P_s^\gamma X_\gamma^0)^{\mathbf{b}}(s)\right)Z_\gamma^{:\mathbf{c}:}(s)\right\|_{\mathcal{C}^{-\nu}} \\
&\quad \leq \left\|v_\gamma(s)\right\|_{(\mathcal{C}^{\frac{1}{2}})^m}^{|\mathbf{a}|-1} s^{-(|\mathbf{b}|-1)(2\nu+\kappa)}\left(\left\|X^0\right\|_{(\mathcal{C}^{-\nu})^m}+\left\|X_\gamma^0\right\|_{(\mathcal{C}^{-\nu})^m}\right)^{|\mathbf{b}|-1}\left\|Z_\gamma^{:\mathbf{c}:}(s)\right\|_{\mathcal{C}^{-\nu}}
\end{aligned}$$

where we used [MW17a, Lemma 7.3]. We get

$$\leq Cs^{-(2n-1)(2\nu-\kappa)}\Bigg(\|\bar{v}_\gamma(s)-v_\gamma(s)\|_{(\mathcal{C}^{\frac{1}{2}})^m} + \left\|X^0 - X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m} \\ + \epsilon^{-\kappa}\sup_{|\mathbf{a}|\leq 2n-1}\left\|Z_\gamma^{:\mathbf{a}:}(s) - H_{\mathbf{a}}(Z_\gamma(s),\mathfrak{c}_\gamma(s))\right\|_{L^\infty(\Lambda_\varepsilon)}\Bigg)$$

where the constant depends on ν,κ,n,T, $\left\|X^0\right\|_{\mathcal{C}^{-\nu}}$, $\left\|X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m}$, $\sup_{|\mathbf{a}|\leq 2n-1}\left\|Z_\gamma^{:\mathbf{a}:}\right\|_{\mathcal{C}^{-\nu}}$ as well as on $\|v_\gamma\|_{L^\infty([0,T];(\mathcal{C}^{\frac{1}{2}})^m)}$, $\|\bar{v}_\gamma\|_{L^\infty([0,T];(\mathcal{C}^{\frac{1}{2}})^m)}$. The last term is estimated probabilistically with Proposition 2.4.7, where the supremum on the torus is bounded be the supremum on Λ_ε at a cost of an arbitrarily small negative power of ϵ. Using Proposition B.0.6 we then bound the $\mathcal{C}^{1/2}$ Besov norm of the second term in (2.103) with the sum

$$C\int_0^t (t-s)^{-\frac{1}{4}-\frac{\nu}{2}-\kappa}s^{-\kappa}s^{-(2n-1)(2\nu-\kappa)}\|\bar{v}_\gamma(s)-v_\gamma(s)\|_{(\mathcal{C}^{\frac{1}{2}})^m}\,ds + C\left\|X^0-X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m} \\ + C^2\int_0^t (t-s)^{-\frac{1}{4}-\frac{\nu}{2}-\kappa}s^{-\kappa}s^{-(2n-1)(2\nu-\kappa)}\epsilon^{-2\kappa}\left(\gamma^{1-\kappa-\nu}+\gamma^{-\kappa}s^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\right)^2\gamma^{-1} \\ + \int_0^t (t-s)^{-\frac{1}{4}-\frac{\nu}{2}-\kappa}s^{-\kappa-(2n-1)(2\nu-\kappa)}\epsilon^{-2\kappa}\gamma\sup_{|\mathbf{a}|\leq 2n-1}\frac{\left\|Z_\gamma^{:\mathbf{a}:}(s)-H_{\mathbf{a}}(Z_\gamma(s),\mathfrak{c}_\gamma(s))\right\|^2_{L^\infty}}{\left(\gamma^{1-\kappa-\nu}+\gamma^{-\kappa}s^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\right)^2}ds \tag{2.107}$$

in the last line we used the inequality $CAB \leq \frac{1}{2}(C^2\gamma^{-1}A^2+\gamma B^2)$. The last term is bounded in expectation with Proposition 2.4.7 with $b=\frac{1}{2}$ (note the absence of any multiplicative constant in front of the last term). Using the fact

$$(t-s)^{-\frac{1}{4}-\frac{\nu}{2}-\kappa}s^{-\kappa}s^{-(2n-1)(2\nu-\kappa)} \leq C(T,\kappa,\nu,n)(t-s)^{-\frac{1}{3}}s^{-\frac{1}{6}}$$

for small enough $\kappa,\nu>0$. Collecting together (2.107), (2.105) and (2.104) with $\lambda=\frac{1}{2}$ we obtain the bound

$$\left\|\bar{v}^{(j)}_\gamma(\cdot,t)-v^{(j)}_\gamma(\cdot,t)\right\|_{\mathcal{C}^{\frac{1}{2}}} \leq C_1\int_0^t (t-s)^{-\frac{1}{3}}s^{-\frac{1}{6}}\|\bar{v}_\gamma(\cdot,s)-v_\gamma(\cdot,s)\|_{(\mathcal{C}^{1/2})^m}\,ds \\ + C_1\left(\left\|X^0-X^0_\gamma\right\|_{(\mathcal{C}^{-\nu})^m}+\gamma^{\frac{1}{2}}+\gamma^{\lambda_0-2\kappa}\right)+C_2\overline{\mathsf{Err}}_2(t)$$

and the expectation of

$$
\begin{aligned}
&\sup_{t\le T}|\overline{\mathsf{Err}_2}(t)| \le C(T)\sup_{t\le T}\gamma^{-1}\int_0^t (t-s)^{-\frac{1}{3}}\left\|Z_\gamma^{\mathrm{high}}(\cdot,s)\right\|^2_{(L^\infty)^m} ds\\
&+C(T)\gamma\epsilon^{-\kappa}\sup_{t\le T}\int_0^t (t-s)^{-\frac{1}{3}}s^{-\frac{1}{6}}\sup_{|\mathbf{a}|\le 2n-1}\frac{\left\|Z_\gamma^{:\mathbf{a}:}(s)-H_{\mathbf{a}}(Z_\gamma(s),\mathfrak{c}_\gamma(s))\right\|^2_{L^\infty(\Lambda_\varepsilon)}}{\left(\gamma^{1-\kappa-\nu}+\gamma^{-\kappa}s^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\right)^2}ds
\end{aligned}
$$

is bounded by $C(T,\nu,\mathfrak{m},\kappa,n)(\gamma\epsilon^{-\kappa}+\gamma^{1-\kappa})$ where we used the scaling (2.20). In the above equation the constants are as after (2.102). □

Chapter 3

Tightness of Ising-Kac model in a two dimensional torus

3.1 Introduction

In Chapter 2 we used the techniques in [MW17a] and [SW16] to show the convergence in law of the evolution of the local fluctuation of the magnetization field under the Glauber dynamic for a large family of spin systems. The strategy however is not sufficient to imply the tightness of the sequence of the (static) fluctuation of the magnetic field under the Gibbs measures. In this chapter we return to the classic Ising-Kac model defined in [MW17a] on a periodic two dimensional lattice Λ_N, where each spin takes values in $\{\pm 1\}$ and we will prove the tightness of the sequence of the local fluctuation fields of the magnetization in any Besov space of negative regularity $\mathcal{C}^{-\nu}$ with $\nu > 0$.

Moreover we are also able to characterize the limit as the $\Phi^4(\mathbb{T}^2)$ measure, formally described by

$$\mathcal{Z}^{-1} \exp\left(\int_{\mathbb{T}^2} \frac{1}{2}\Phi(x)\Delta\Phi(x) - \frac{1}{12}\Phi^{:4:}(x) + \frac{A}{2}\Phi^2(x)\, dx \right) d\Phi \; , \tag{3.1}$$

where $\Phi^{:4:}$ is a renormalization of the forth power of the field.

In the process of writing the article [HI18], we came across the work [CMP95], in which the authors showed the convergence of the 2D Ising-Kac model on $\mathbb{Z}^2$ to Φ^4_2 by proving the convergence of the discrete Schwinger functions. In particular they were the first (to our knowledge) to explain the small shift of the critical temperature for the Ising-Kac model with the renormalisation constants of the Wick powers. The result (see [CMP95, Theorem 2]) is restricted to temperatures satisfying certain technical condition that allows the use of Aizenman's correlation inequalities.

Our result resembles the one obtained in [CMP95], with some differences. We will work on a periodic lattice instead of $\mathbb{Z}^2$, which we think of as a discretisation of a 2D torus. This restriction is mainly due to our techniques for bounding the solutions globally in time and a posteriori doesn't appear to be strictly necessary since the limiting dynamic can be defined also on the whole 2D plane (see [MW17b]). Moreover, as our proof exploits the dissipativity of the Glauber dynamic and not the correlation inequalities, we do not have the restriction on the temperature present in [CMP95, Theorem 2], so that we cover arbitrary values $A \in \mathbb{R}$ in (3.1). A correlation inequality, the GHS inequality, is then employed in a subsequent corollary to partially extend the result to the case of arbitrary external magnetization b. Corollary 3.2.3 is the only place where we use a correlation inequality.

Our main result in this chapter is Theorem 3.2.1 showing tightness of the fluctuations of local averages of the magnetic field in any Besov space of negative regularity. The proof of the main result in Theorem 3.2.1 is based on the analysis of the dynamical Φ^4_2 model in [TW16, Sec. 3] and makes no use of correlation inequalities (not explicitly at least), avoids the restriction (1.8) of [CMP95] and exploits the regularisation provided by the time evolution of the Glauber dynamic. As a consequence of Theorem 3.2.1, we obtain in Corollary 3.2.2 tightness in $\mathcal{S}'(\mathbb{T}^2)$ for the fluctuation fields, appropriately rescaled (see also Corollary 3.2.3 in case of a Gibbs measure with an external magnetization).

In Theorem 3.2.4 we characterise the limit of each subsequence to be an invariant measure for the dynamical Φ^4_2 model constructed in [DPD03]. Since it was shown in [DPD03] that (3.1) is such a measure and in [TW16] that this invariant measure is unique, the result follows. For the proof, we make use of the uniform convergence to the invariant measure and the convergence of the Glauber dynamic in [MW17a].

The result described in this chapter is a joint work with M. Hairer and has been published the Journal of Statistical Physics [HI18].

3.2 Statements of the theorems

For N be a positive integer consider $\Lambda_N = \{1-N, \dots, N\}^2$, $b \in \mathbb{R}$, and σ is a configuration belonging to $\{-1, 1\}^{\Lambda_N}$. As introduced in Chapter 1 we will also consider the discretization of the two dimensional torus $\mathbb{T}^2$ of mesh ϵ, denoted by $\Lambda_\varepsilon = (\epsilon\mathbb{Z})^2 \cap [-1, 1]^2$. Consider the Hamiltonian

$$\mathcal{H}^\gamma_\beta(\sigma) = \frac{\beta}{2} \sum_{x,y\in\Lambda_N} \kappa_\gamma(x, y)\sigma_x\sigma_y \tag{3.2}$$

and recall the definition of the Gibbs measure in (1.7) over Σ_N associated to the potential (1.5), with inverse temperature β and external magnetic field b

$$\mu_{\gamma,b}(\sigma) \stackrel{\text{def}}{=} \left(\mathscr{Z}^N_{\gamma,\beta,b}\right)^{-1} \exp\left\{\mathcal{H}^\gamma_\beta(\sigma) + b\sum_{x\in\Lambda_N}\sigma_x\right\} \tag{3.3}$$

where $\mathscr{Z}^N_{\gamma,\beta,b}$ is the partition function. When $b=0$ we will use the notation μ_γ instead of $\mu_{\gamma,0}$. Recall moreover, for $x\in\Lambda_N$, the definition of the local average of the spins

$$h_\gamma(x) \stackrel{\text{def}}{=} \sum_{z\in\Lambda_N}\kappa_\gamma(x-z)\sigma_z \tag{3.4}$$

Assume for the moment that the external magnetization, denoted with b in (3.2) is equal to zero, which is also the case studied in [MW17a]. Following [BPRS93, MW17a], we define the magnetisation fluctuation field over the lattice Λ_ε as $X_\gamma(z) = \gamma^{-1}h_\gamma(\epsilon^{-1}z)$. We will consider a dynamic of Glauber type on Σ_N in order to gain insight into the properties of the fluctuations. In order for this dynamic to converge to a non-trivial limit, we will enforce the relation between the scalings ϵ and γ given by (3.25).

The dynamic can be described informally as follows. Each site $x\in\Lambda_N$ is assigned an independent exponential clock with rate 1. When the clock rings, the corresponding spin changes sign with probability

$$c_\gamma(z,\sigma) = \frac{1}{2}\left(1-\sigma_z\tanh\left(\beta h_\gamma(z)\right)\right)\ , \tag{3.5}$$

and remains unchanged otherwise. More formally, the generator of this dynamic is given by

$$\mathscr{L}_\gamma f(\sigma) = \sum_{z\in\Lambda_N} c_\gamma(z,\sigma)\left(f(\sigma^{\{z\}})-f(\sigma)\right)\ , \tag{3.6}$$

for $f:\Sigma_N\to\mathbb{R}$, where

$$\sigma^{\{z\}}_y = \begin{cases} -\sigma_z & \text{if } y=z, \\ \sigma_y & \text{if } y\neq z. \end{cases}$$

The probabilities $c_\gamma(z,\sigma)$ are chosen precisely in such a way that μ_γ is invariant for this Markov process. We shall use the notations $\sigma_x(s)$ and $h_\gamma(x,s)$ to refer to the process at (microscopic) space $x\in\Lambda_N$ and time $s\in\mathbb{R}_+$. In order to rewrite the process in macroscopic coordinates, we speed up the generator $\mathscr{L}_\gamma$ by a factor α^{-1} and apply it to

$$X_\gamma(x,s) \stackrel{\text{def}}{=} \gamma^{-1}h_\gamma(\epsilon^{-1}x,\alpha^{-1}s)\ ,$$

in (macroscopic) space $x \in \Lambda_\varepsilon$ and time $s \in \mathbb{R}_+$. In [MW17a, Theorem 3.2] it is proven that, if the parameters α, ϵ and the inverse temperature β are chosen such that

$$\alpha = \gamma^2 , \qquad \epsilon = \gamma^2 , \quad \beta - 1 = \alpha \left(\mathfrak{c}_\gamma + A \right) , \tag{3.7}$$

where $\mathfrak{c}_\gamma$ is described in 3.10 the law of X_γ on $\mathcal{D}(\mathbb{R}_+, \mathcal{C}^{-\nu})$, converges in distribution to the solution of the stochastic quantization equation

$$\partial_t X = \Delta X - \frac{1}{3} X^{:3:} + AX + \sqrt{2}\xi , \qquad X(\cdot, 0) = X^0 \in \mathcal{C}^{-\nu} \tag{3.8}$$

where $X_\gamma(\cdot, 0) \to X^0$ in $\mathcal{C}^{-\nu}$ and ξ is a space time white noise and the expression $X^{:3:}$ stands for a renormalized power defined in [DPD03], where the relevant notion of "solution" to (3.8) is also given. The solution theory of (3.8) has been briefly summarised in Subsection 2.2.6. For $x \in \Lambda_\varepsilon$, recall $K_\gamma(x) = \epsilon^{-2} \kappa_\gamma(\epsilon^{-1} x)$, the macroscopic version of the kernel K_γ, already used in Chapter 2 and define the discrete Laplacian $\Delta_\gamma f = \epsilon^{-2}\gamma^2 (K_\gamma * f - f)$. Under the Glauber dynamic, the process X_γ satisfies on $\Lambda_\varepsilon \times [0, T]$

$$\begin{aligned} X_\gamma(x,t) &= X_\gamma(x,0) \\ &\quad + \int_0^t \Delta_\gamma X_\gamma(x,s) - \frac{1}{3} \left(X_\gamma^3(x,s) - \mathfrak{c}_\gamma X_\gamma(x,s) \right) + A X_\gamma(x,s) \, ds \\ &\quad + \int_0^t \mathcal{O}(\gamma^2 X_\gamma^5(x,s)) \, ds + M_\gamma(x,t) \end{aligned} \tag{3.9}$$

where $M_\gamma(x,t)$ is a martingale and $\mathfrak{c}_\gamma$ is the logarithmically diverging constant

$$\mathfrak{c}_\gamma = \frac{1}{4} \sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{|\hat{K}_\gamma(\omega)|^2}{\epsilon^{-2}\gamma^2 (1 - \hat{K}_\gamma(\omega))} . \tag{3.10}$$

Following the analysis of Chapter 2, we will consider the decomposition of $X_\gamma = Z_\gamma + V_\gamma$. Where Z_γ solves the linearized part of the equation

$$\begin{cases} Z_\gamma(x,t) &= \Delta_\gamma Z_\gamma(x,t) + dM_\gamma(x,t) \\ Z_\gamma(x,0) &= 0 \end{cases} \tag{3.11}$$

From (3.9) and (3.11) we see that $V_\gamma(x,t)$ satisfies, for $x \in \Lambda_\varepsilon, t \geq 0$

$$V_\gamma(x,t) = V_\gamma^0(x) + \int_0^t \Delta_\gamma V_\gamma(x,s) + \gamma^{-2} K_\gamma * \left(\gamma^{-1} \tanh(\beta \gamma X_\gamma(s)) - X_\gamma(s) \right)(x) ds$$

and in particular $V_\gamma(x,t)$ it is differentiable in time. In Subsection 1.2.2 we introduced two different notions of Besov norm for functions defined on the discretized torus Λ_ε that we are going to resume now. Firstly recall the definition of the extension operator given in (1.9). For $f : \Lambda_\varepsilon \to \mathbb{R}$

$$\mathsf{Ext}(f)(x) \stackrel{\text{def}}{=} \frac{1}{2^d} \sum_{\omega \in \Lambda_N^d} \widehat{f}(\omega) e^{\pi i w \cdot x} \qquad \text{for } x \in \mathbb{T}^d$$

where $\widehat{f}(\omega)$ is the discrete Fourier transform of f. Recall the definition (1.10) of the *continuous Besov norm* or simply *Besov norm* $\|\cdot\|_{\mathcal{B}^\nu_{p,q}}$ with regularity $\nu \in \mathbb{R}$ and parameters $p, q \in [1, \infty]$, applied to $f : \Lambda_\varepsilon \to \mathbb{R}$

$$\|f\|_{\mathcal{B}^\nu_{p,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k \mathsf{Ext}(f)\|^q_{L^p(\mathbb{T}^d)}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \|\delta_k \mathsf{Ext}(f)\|_{L^p(\mathbb{T}^d)} & \text{if } q = \infty \end{cases}$$

and $\|\cdot\|_{\mathcal{C}^\nu} = \|\cdot\|_{\mathcal{B}^\nu_{\infty,\infty}}$. This norm has been used in Chapter 2 and in [MW17a]. One of the reason that makes the Besov norm useful is that for any $T > 0$, $\nu > 0$ and $\lambda > 0$,

$$\limsup_{\gamma \to 0} \mathbb{E}\left[\sup_{s \in [0,T]} s^\lambda \|H_n(Z_\gamma(\cdot, s), \mathfrak{c}_\gamma)\|_{\mathcal{C}^{-\nu}}\right] < \infty , \tag{3.12}$$

which follows from Propositions 5.3 and 5.4 and [MW17a, Eq. 3.15]. In Proposition 3.3.3 we will need however the discrete notion of the above norm, defined in (1.12), which we called *discrete Besov norm* tailored for functions defined on the discretized torus.

$$\|f\|_{\mathcal{B}^\nu_{p,q}(\Lambda^d_\varepsilon)} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k \mathsf{Ext}(f)\|^q_{L^p(\Lambda^d_\varepsilon)}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda^d_\varepsilon)} & \text{if } q = \infty \end{cases}$$

and similarly $\|\cdot\|_{\mathcal{C}^\nu(\Lambda^d_\varepsilon)} = \|\cdot\|_{\mathcal{B}^\nu_{\infty,\infty}(\Lambda^d_\varepsilon)}$. The next theorem is the main result of the chapter.

Theorem 3.2.1 *Assume $b = 0$. Then for all positive $\nu > 0$ and for all $p > 0$*

$$\limsup_{\gamma \to 0} \mu_\gamma \left[\|X_\gamma\|^p_{\mathcal{C}^{-\nu}}\right] < \infty .$$

In particular, the laws of X_γ form a tight set of probability measures on $\mathcal{C}^{-\nu}$.

From the above theorem it is possible to deduce the following Corollary, where we extended the discrete spin field to the continuous torus using piecewise constant functions.

Corollary 3.2.2 *Assume $b = 0$. Then the law of the field $\left(\gamma^{-1}\sigma_{\lfloor \epsilon^{-1} x \rfloor}\right)_{x \in \mathbb{T}^2}$ under μ_γ is tight in $\mathcal{S}'(\mathbb{T}^2)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{T}^2)$ and consider

$$\left\langle \gamma^{-1}\sigma_{\lfloor \epsilon^{-1}\cdot \rfloor}, \varphi \right\rangle_{\mathbb{T}^2} = \sum_{x\in\Lambda_N} \epsilon^2 \left(\gamma^{-1}\sigma_x\right) \bar{\varphi}(\epsilon x)$$

where $\bar{\varphi}(\epsilon x) = \epsilon^{-2}\int_{|y|_\infty \le 2^{-1}\epsilon} \varphi(\epsilon x + y)\, dy$. Using the differentialbility of φ, we replace $\bar{\varphi}$ with $\kappa_\gamma * \bar{\varphi}$ at the cost of

$$\epsilon^2\gamma^{-2} \sup_{i_1,i_2\in\{1,2\}} \|\partial_{i_1}\partial_{i_2}\varphi\|_{L^\infty(\mathbb{T}^2)} = \mathcal{O}(\gamma^2)\ .$$

Therefore

$$\left\langle \gamma^{-1}\sigma_{\lfloor \epsilon^{-1}\cdot \rfloor}, \varphi \right\rangle_{\mathbb{T}^2} = \langle X_\gamma, \bar{\varphi}\rangle_{\Lambda_\varepsilon} + \mathcal{O}(\gamma) = \langle \mathsf{Ext} X_\gamma, \varphi\rangle_{\mathbb{T}^2} + \mathcal{O}(\gamma)$$

the corollary follows from Theorem 3.2.1. □

We now show how to extend the previous result to the case $b \neq 0$. It is clear that, by symmetry it is sufficient to assume $b \geq 0$. In case of ferromagnetic pair potential $\kappa_\gamma \geq 0$ with positive external magnetisation $b \geq 0$, the following inequality holds

$$\mu_{\gamma,b}\left[\sigma_x;\sigma_y\right] \leq \mu_{\gamma,b}\left[\sigma_x;\sigma_y\right] \tag{3.13}$$

where $\mu_{\gamma,b}\left[\sigma_x;\sigma_y\right]$ is the covariance between the spins. The above inequality follows from the fact that $\frac{d}{db}\mu_{\gamma,b}\left[\sigma_x;\sigma_y\right] \leq 0$ which is an immediate consequence of the GHS inequality (see for instance [Leb74] for a proof), valid for $\kappa_\gamma \geq 0$ and $b \geq 0$.

Corollary 3.2.3 *Consider any map $\gamma \mapsto b_\gamma \geq 0$ and denote by $m_\gamma(b) = \mu_{\gamma,b}[\sigma_x]$ the mean of the spin σ_x, which is independent of $x \in \Lambda_N$. Then the law of the field*

$$\left(\frac{\sigma_{\lfloor \epsilon^{-1}x\rfloor} - m_\gamma(b_\gamma)}{\gamma}\right)_{x\in\mathbb{T}^2}$$

under $\mu_{\gamma,b}$ is tight in $\mathcal{S}'(\mathbb{T}^2)$.

Proof. Fixing a test function φ and replacing $\bar{\varphi}$ with $\kappa_\gamma * \bar{\varphi}$ as in Corollary 3.2.2, we can assume to have

$$\sum_{x\in\Lambda_N} \epsilon^2 \left(\frac{\sigma_x - m_\gamma(b_\gamma)}{\gamma}\right) \kappa_\gamma * \bar{\varphi}(\epsilon x) + \mathcal{O}(\gamma)\ .$$

Decompose $\varphi = \varphi^+ - \varphi^-$ into its positive and negative part. For each of them, using the correlation inequality (3.13), we have that

$$\mu_{\gamma,b}\Big|\epsilon^2 \sum_{x\in\Lambda_N} \gamma^{-1}\left(\sigma_x - m_\gamma(b_\gamma)\right)\kappa_\gamma * \overline{\varphi^{\pm}}(\epsilon x)\Big|^2 \le \mu_\gamma\Big|\epsilon^2 \sum_{x\in\Lambda_N} \left(\gamma^{-1}\sigma_x\right)\kappa_\gamma * \overline{\varphi^{\pm}}(\epsilon x)\Big|^2$$

Using Theorem 3.2.1 we see that, for $\nu \in (0,1)$, this quantity is bounded uniformly by a fixed multiple of $\|\varphi^{\pm}\|^2_{\mathcal{B}^{\nu}_{1,1}}\,\mu_\gamma\big[\,\|X_\gamma\|^2_{\mathcal{C}^{-\nu}}\,\big]$, up to an error of order $\mathcal{O}(\gamma)$. In order to conclude we observe that, for $\nu \in (0,1)$

$$\left\|\varphi^{\pm}\right\|_{\mathcal{B}^{\nu}_{1,1}} \lesssim \left\|\varphi^{\pm}\right\|_{L^1} + \left\|\varphi^{\pm}\right\|^{\nu}_{\text{Lip}}\left\|\varphi^{\pm}\right\|^{1-\nu}_{L^1} \lesssim \|\varphi\|_{L^1} + \|\nabla\varphi\|_{L^\infty}$$

where the first inequality is (A.9), generalised to Lipschitz functions. □

The next theorem provides a characterisation for the limit of the subsequences, only in the symmetric case $b = 0$.

Theorem 3.2.4 *Assume $b = 0$, then under μ_γ any limiting law of the sequence $\{X_\gamma\}_\gamma$ coincides with the unique invariant measure for the dynamic 3.8, and hence, by [TW16] and [DPD03, Remark 4.3], coincides with the $\Phi^4(\mathbb{T}^2)$ measure.*

Proof. For the sake of precision we will explicitly write $\mathsf{Ext}(X_\gamma(t))$ where the process X_γ has been extended to the whole torus.

We will use the Glauber dynamic and the solution of the stochastic quantisation equation (3.8) introduced in the previous section: the idea is to exploit the exponential convergence to the invariant measure of the solution of the SPDE (3.8) proved in [TW16] and the convergence of the Glauber dynamic of the Kac-Ising model in [MW17a].

By [MW17a, Thm 3.2], we know that if for $0 < \kappa < \nu$ small enough the sequence of initial conditions $\mathsf{Ext}(X^0_\gamma)$ is bounded in $\mathcal{C}^{-\nu+\kappa}$ and converges to a limit X^0 in $\mathcal{C}^{-\nu}$ as $\gamma \to 0$, one has

$$\mathsf{Ext}\,(X_\gamma) \xrightarrow{\mathcal{L}} X \qquad \text{in } \mathcal{D}\left([0,T];\mathcal{C}^{-\nu}\right) \tag{3.14}$$

where X solves (3.8) starting from X^0. In the above equation we took into account the fact that X_γ is defined on the discrete lattice and therefore has to be extended with the operator Ext to be comparable with X.

We first want to show that (3.14) holds true when instead of a deterministic sequence bounded in $\mathcal{C}^{-\nu+\kappa}$ and $\mathsf{Ext}X^0_\gamma \to X^0$ in $\mathcal{C}^{-\nu}$, we have the convergence in law of the initial conditions $\mathcal{L}(\mathsf{Ext}X^0_\gamma) \to \mathcal{L}(X^0)$ in the topology of $\mathcal{C}^{-\nu}$ and tightness in $\mathcal{C}^{-\nu+\kappa}$. In order to do this call $\mathfrak{L}_\gamma$ (resp. $\mathfrak{L}_0$) the laws at time zero of the processes $\mathsf{Ext}X_\gamma$ (resp. X) and assume that $\mathfrak{L}_\gamma \to \mathfrak{L}_0$ with respect to the topology of $\mathcal{C}^{-\nu}$ and $\mathfrak{L}_\gamma$ is tight in $\mathcal{C}^{-\nu+\kappa}$. Consider then a bounded continuous function $G : \mathcal{D}\left([0,T];\mathcal{C}^{-\nu}\right) \to \mathbb{R}$: we want to show that

$$\lim_{\gamma\to 0} \left|\mathbb{E}\big[G(\mathsf{Ext}(X_\gamma))\big|X^0_\gamma \sim \mathfrak{L}_\gamma\big] - \mathbb{E}\big[G(X)\big|X^0 \sim \mathfrak{L}_0\big]\right| = 0\,.$$

Conditioning over the initial conditions we can define

$$\begin{aligned} f^\gamma_G(X^0_\gamma) &:= \mathbb{E}\left[G(\mathsf{Ext}(X_\gamma))\middle|X_\gamma(0) = X^0_\gamma\right] \\ f_G(X^0) &:= \mathbb{E}\left[G(X)\middle|X(0) = X^0\right]\,. \end{aligned}$$

The result [MW17a, Thm 3.2] implies that $f^\gamma_G(X^0_\gamma) \to f_G(X^0)$ whenever $\mathsf{Ext}X^0_\gamma \to X^0$ in $\mathcal{C}^{-\nu}$ and $\limsup_{\gamma\to 0}\left\|\mathsf{Ext}X^0_\gamma\right\|_{\mathcal{C}^{-\nu}} < \infty$. Since $\mathcal{C}^{-\nu}$ is separable, we can apply the Skorokhod's representation theorem to deduce that there is a probability space $(\tilde{\mathbb{P}}, \tilde{\mathcal{F}}, \tilde{\Omega})$ where all the processes $\mathsf{Ext}(X^0_\gamma)$ and X^0 can be realised and the sequence $\mathsf{Ext}(X^0_\gamma)(\tilde\omega)$ converge to $X^0(\tilde\omega)$ in $\mathcal{C}^{-\nu}$ for $\tilde{\mathbb{P}}$-a.e. $\tilde\omega \in \tilde\Omega$.
Therefore an application of the dominated convergence theorem implies that, as $\gamma \to 0$

$$\begin{aligned} &\left|\mathbb{E}\big[G(\mathsf{Ext}(X_\gamma))\big|X^0_\gamma \sim \mathfrak{L}_\gamma\big] - \mathbb{E}\big[G(X)\big|X^0 \sim \mathfrak{L}_0\big]\right| \\ &\qquad\qquad\qquad \le \int \left|f^\gamma_G(X^0_\gamma(\tilde\omega)) - f_G(X^0(\tilde\omega))\right| \tilde{\mathbb{P}}(d\tilde\omega) \to 0\,. \end{aligned} \tag{3.15}$$

And therefore we can assume (3.14) to hold even when the initial datum is convergent in law.

By Theorem 3.2.1 we know that, if at time 0 the configuration $\sigma(0) \in \Sigma_N$ is distributed according to μ_γ, then the law of $X^0_\gamma(x) = \gamma^{-1}\kappa_\gamma * \sigma_{\lfloor\epsilon^{-1}x\rfloor}(0)$ is tight, and therefore there exists a subsequence γ_k for $k \ge 0$ and a measure μ^* on $\mathcal{C}^{-\nu}$ such that the law of $\mathsf{Ext}X^0_{\gamma_k}$ converges to μ^*. In the following calculations we will tacitly assume $\gamma \to 0$ along the sequence γ_k to avoid the subscript. We will show that, if μ if the unique invariant measure of (3.8) then $\mu^* = \mu$.

Let $F : \mathcal{C}^{-\nu} \to \mathbb{R}$ be a bounded and continuous function, then, by the stationary of the Gibbs measure for the Glauber dynamic, for $t \ge 0$

$$\mathbb{E}^\gamma_\beta\left[F(\mathsf{Ext}X_\gamma(0))\right] = \mathbb{E}^\gamma_\beta\left[F(\mathsf{Ext}X_\gamma(t))\right]\,.$$

Recall that the evaluation map, that associates to a process in $\mathcal{D}\left([0,T];\mathcal{C}^{-\nu}\right)$ its value at a given time, is not continuous with respect to the Skorokhod topology, however the integral map $G : u \mapsto \int_0^T F(u(s))\,ds$ is continuous in its argument in virtue of the the continuity and boundedness of F. Hence for any fixed T we have

$$\mathbb{E}_\beta^\gamma \left[F(\mathsf{Ext}X_\gamma(0))\right] = \mathbb{E}_\beta^\gamma\left[T^{-1}\int_0^T F\left(\mathsf{Ext}X_\gamma(s)\right)\,ds\right]$$

and

$$\lim_{\gamma\to 0}\left|\mathbb{E}_\beta^\gamma\left[\int_0^T F\left(\mathsf{Ext}X_\gamma(s)\right)\,ds\right] - \mathbb{E}\left[\int_0^T F\left(X(s)\right)\,ds\middle|X(0)\sim\mu^*\right]\right| = 0\;.$$

By the uniform convergence to equilibrium of the stochastic quantisation equation [TW16, Cor. 6.6] there exist constants $c, C > 0$

$$\left|\mathbb{E}[F\big(X(s)\big)\big|X(0)\sim\mu^*] - \mu[F]\right| \le C\,|F|_\infty\, e^{-cs}\;.$$

From the above inequality it follows that

$$\left|T^{-1}\int_0^T \mathbb{E}[F\big(X(s)\big)\big|X(0)\sim\mu^*] - \mu[F]ds\right| \lesssim T^{-1}\,|F|_\infty$$

and letting T be large enough the last difference can be made arbitrarily small. From the above estimates we can see that, for arbitrary $T > 0$,

$$\limsup_{\gamma\to 0}\left|\mathbb{E}_\beta^\gamma\left[F(\mathsf{Ext}X_\gamma(0))\right] - \mu[F(X)]\right| \le C\,|F|_\infty\,T^{-1}$$

and the result follows. □

3.3 Proof of Theorem 3.2.1

We will now prove the statements used in Section 3.2. The statement of the next proposition doesn't describe the correct behaviour of the process X_γ, however it can be used as a starting point for the derivation of more precise bounds.

Proposition 3.3.1 *Let $p \ge 2$ an even integer, and $\lambda \in [0,1]$ then there exists $C(p,\lambda) > 0$ such that*

$$\mathbb{E}\big[\,\|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)}\big] \le C\left(\mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\big]^{1-\lambda} t^{-\frac{p}{2}\lambda}\right) \vee \gamma^{-\frac{p}{2}}\;.$$

In particular, if we start the process from the invariant measure, we obtain that there exists

$C = C(p) > 0$ *such that for all* $t \geq 0$

$$\mathbb{E}_{\mu_\gamma}\big[\, \|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)} \big] \leq C(p)\gamma^{-\frac{p}{2}} \tag{3.16}$$

Proof. In the following proof we will denote with C a generic constant whose value depends on p and might change from line to line. Recall the action of the generator of the Glauber dynamic (3.6):

$$\begin{aligned}\mathscr{L}_\gamma h^p_\gamma(t,x) &= \sum_{z\in\Lambda_N} c_\gamma(z,\sigma(t))\Big((h_\gamma(t,x) - 2\sigma_z(t)\kappa_\gamma(z-x))^p - h^p_\gamma(t,x)\Big) \\ &\leq p\big(-h_\gamma + \kappa_\gamma * \tanh(\beta h_\gamma)\big)(t,x)h^{p-1}_\gamma(t,x) + C\big(|h_\gamma(t,x)| + \gamma^2\big)^{p-2}\gamma^2\,.\end{aligned}$$

The second inequality is a consequence of the fact that $\|\kappa_\gamma\|_\infty \lesssim \gamma^2$ and $\|\kappa_\gamma\|_{L^1} \lesssim 1$. We can take the average over $x \in \Lambda_N$ to obtain

$$\begin{aligned}\mathscr{L}_\gamma \|h_\gamma(t)\|^p_{L^p(\Lambda_N)} \leq p\,\big\langle h^{p-1}_\gamma(t), \kappa_\gamma * \tanh(\beta h_\gamma(t))\big\rangle_{\Lambda_N} - p\,\|h_\gamma(t)\|^p_{L^p(\Lambda_N)} \\ + C\gamma^2\,\|h_\gamma(t)\|^{p-2}_{L^{p-2}(\Lambda_N)} + C\gamma^{2p-2}\,.\end{aligned}$$

We use the fact that p is even and the hyperbolic tangent is monotone to bound

$$\begin{aligned}&\big\langle h^{p-1}_\gamma(t), \kappa_\gamma * \tanh(\beta h_\gamma(t))\big\rangle_{\Lambda_N} = \big\langle h^{p-1}_\gamma(t), \tanh(\beta h_\gamma(t))\big\rangle_{\Lambda_N} \\ &+ \frac{1}{2}\sum_{x,y\in\Lambda_N} \kappa_\gamma(x-y)\big(h^{p-1}_\gamma(t,x) - h^{p-1}_\gamma(t,y)\big)\big(\tanh(\beta h_\gamma(t,y)) - \tanh(\beta h_\gamma(t,x))\big) \\ &\qquad\qquad \leq \big\langle h^{p-1}_\gamma(t), \tanh(\beta h_\gamma(t))\big\rangle_{\Lambda_N}\,.\end{aligned}$$

Moreover, it is easy to see that there exists a constant $c_0 > 0$ such that

$$\frac{\tanh(\beta h)}{h} \leq \beta - c_0 h^2 \qquad \text{for } h \in [1,1]\,.$$

Since $|h_\gamma(t,x)| \leq 1$ and $\beta = 1 + \gamma^2(\mathfrak{c}_\gamma + A)$, we can bound $\mathscr{L}_\gamma \|h_\gamma(t)\|^p_{L^p(\Lambda_N)}$ by

$$\begin{aligned}p[\beta-1]\,\|h_\gamma(t)\|^p_{L^p(\Lambda_N)} &- c_0 p\,\|h_\gamma(t)\|^{p+2}_{L^{p+2}(\Lambda_N)} + C\gamma^2\,\|h_\gamma(t)\|^{p-2}_{L^{p-2}(\Lambda_N)} + C\gamma^{2p-2} \\ &\leq C(\gamma^2\mathfrak{c}_\gamma)^{\frac{p+2}{2}} - \frac{c_0}{2}p\,\|h_\gamma(t)\|^{p+2}_{L^p(\Lambda_N)} + C\gamma^{\frac{p+2}{2}} + C\gamma^{2p-2} \\ &\leq -\frac{c_0}{2}p\,\|h_\gamma(t)\|^{p+2}_{L^p(\Lambda_N)} + C\gamma^{\frac{p+2}{2}}\,,\end{aligned}$$

where we used the fact that $|A| \leq \mathfrak{c}_\gamma \leq \gamma^{-1}$ for γ small enough and the generalised Young inequality in the last line. Therefore, taking the expectation

$$\begin{aligned}
&\mathbb{E}\big[\,\big\|h_\gamma(\alpha^{-1}t)\big\|^p_{L^p(\Lambda_N)}\,\big] \\
&\qquad \leq \mathbb{E}\big[\,\|h_\gamma(0)\|^p_{L^p(\Lambda_N)}\,\big] + \alpha^{-1}\int_0^t -\frac{c_0}{2}p\mathbb{E}\big[\,\big\|h_\gamma(\alpha^{-1}s)\big\|^{p+2}_{L^p(\Lambda_N)}\,\big] + C\gamma^{\frac{p+2}{2}}ds
\end{aligned}$$

and multiplying both sides by γ^{-p} and applying Jensen's inequality we obtain

$$\begin{aligned}
\mathbb{E}\big[\,\|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] &= \mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] + \int_0^t \mathbb{E}\big[\mathscr{L}_\gamma\,\|X_\gamma(s)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]\,ds \\
&\leq \mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] - \frac{c_0}{2}p\gamma^2\int_0^t \mathbb{E}\big[\,\|X_\gamma(s)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]^{\frac{p+2}{p}}\,ds + C\gamma^{\frac{2-p}{2}}\;.
\end{aligned}$$

From the comparison test in Lemma A.0.12 we have that

$$\mathbb{E}\big[\,\|X_\gamma(t)\|^p_{L^p(\Lambda_\varepsilon)}\,\big] \lesssim \frac{\mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p}\,\big]}{\big(1+Ct\mathbb{E}\big[\,\|X_\gamma(0)\|^p_{L^p(\Lambda_\varepsilon)}\,\big]^{\frac{2}{p}}\big)^{\frac{p}{2}}} \vee \gamma^{-\frac{p}{2}}\;,$$

and the result follows. □

Remark 3.3.2 Despite its simplicity, Proposition 3.3.1 has the advantage of making the proof of [MW17a, Theorem 6.1] simpler, avoiding the introduction of the stopping time $\tau_{\gamma,\mathfrak{m}}$ and providing a sufficient control over [MW17a, Eq. 6.7].

Proposition 3.3.3 *Recall the definition of the processes X_γ, Z_γ and V_γ given in the Section 3.2. Let $p \geq 2$ an even integer. Then there exist $\nu_0 > 0$, $\lambda_{j,i} > 0$ for $i = 1, 2$ and $j = 0, 1, 2$ such that for all $0 < \nu < \nu_0$ and $0 \leq s \leq t \leq T$*

$$\begin{aligned}
&\|V_\gamma(t,\cdot)\|^p_{L^p(\Lambda_\varepsilon)} - \|V_\gamma(s,\cdot)\|^p_{L^p(\Lambda_\varepsilon)} \\
&\qquad + C_1\int_s^t \|V_\gamma(r,\cdot)\|^{p+2}_{L^p(\Lambda_\varepsilon)}\,dr + C_1\int_s^t \big\langle V_\gamma^{p-1}(r), (-\Delta_\gamma)V_\gamma(r)\big\rangle_{\Lambda_\varepsilon}\,dr \\
&\qquad\qquad \leq C_2\int_s^t \sum_{j=0}^{3}\sum_{i=1,2} \|H_j(Z_\gamma(r,\cdot),\mathfrak{c}_\gamma)\|^{\lambda_{j,i}}_{\mathcal{C}^{-\nu}(\Lambda_\varepsilon)}\,dr + \int_s^t \mathsf{Err}(r)\,dr
\end{aligned} \tag{3.17}$$

where, for every $q > 0$

$$\sup_{0\leq r\leq T} \mathbb{E}\left[\mathsf{Err}^q(r)\right]^{\frac{1}{q}} \lesssim C_3(p,q,T)\gamma^{\frac{p-2}{6}-2\nu\frac{(p-2)}{3}} \tag{3.18}$$

Proof. The proof follows the argument in [TW16, Proposition 3.7], with the important difference that in our case all the operators are discrete operators. Without loss of generality,

we will prove (3.17) starting at time $s=0$ from $V_\gamma^0 = X_\gamma^0$.
In the following calculations, since there is no possibility of confusion, we will use L^p instead of $L^p(\Lambda_\varepsilon)$, and $\langle \cdot,\cdot \rangle$ instead of $\langle \cdot,\cdot\rangle_{\Lambda_\varepsilon}$.
From (3.9) and (3.11) we see that $V_\gamma(x,t)$ satisfies, for $x \in \Lambda_\varepsilon, t \geq 0$

$$\begin{aligned}
&V_\gamma(x,t) \\
&= V_\gamma(x,0) + \int_0^t \Delta_\gamma V_\gamma(x,s) ds + \int_0^t \gamma^{-2} K_\gamma * \left(\gamma^{-1}\tanh(\beta\gamma X_\gamma(s)) - X_\gamma(s)\right)(x)\, ds
\end{aligned}$$

where $V_\gamma(\cdot,0) = X_\gamma(\cdot,0)$ and in particular $V_\gamma(x,t)$ is continuous and weakly differentiable in time, for all $\gamma > 0$. Recall that $\beta = 1+\gamma^2(\mathfrak{c}_\gamma + A)$ and expand the hyperbolic tangent up to third order

$$\begin{aligned}
&\tanh(\beta\gamma X_\gamma(s)) \\
&= \gamma X_\gamma(s) + \gamma^3(\mathfrak{c}_\gamma + A)X_\gamma(s) - \frac{\gamma^3}{3}X_\gamma^3(s) + \gamma^3(\beta-1)\mathcal{O}\left(X_\gamma^3(s)\right) + \mathcal{O}\left(\gamma^5 X_\gamma^5(s)\right).
\end{aligned}$$

With the above formula the derivative of the discrete L^p norm of V_γ is calculated

$$\|V_\gamma(t)\|_{L^p}^p = \left\|V_\gamma^0\right\|_{L^p}^p + p\int_0^t \left\langle V_\gamma^{p-1}, \Delta_\gamma V_\gamma\right\rangle(s)\, ds + \frac{1}{3}D(s) + B(s)\, ds \tag{3.19}$$

where

$$D(s) = -\left\langle K_\gamma * V_\gamma^{p-1}(s), X_\gamma^3(s) - 3(\mathfrak{c}_\gamma + A)X_\gamma(s)\right\rangle$$

and $B(s)$ is produced by the remainder of the Taylor expansion of the hyperbolic tangent

$$B(s) \leq C\gamma^2\left\langle |V_\gamma^{p-1}|(s), \mathfrak{c}_\gamma |X_\gamma|^3(s) + |X_\gamma|^5(s)\right\rangle\ . \tag{3.20}$$

where we used the fact that $|A| \leq \mathfrak{c}_\gamma$ for γ small enough. We will first replace $D(s)$ with

$$\begin{aligned}
D_1(s) &:= -\left\langle V_\gamma^{p-1}(s), X_\gamma^3(s) - 3\mathfrak{c}_\gamma X_\gamma(s)\right\rangle + 3A\left\langle V_\gamma^{p-1}(s), X_\gamma(s)\right\rangle \qquad (3.21)\\
&\leq -\left\|V_\gamma^{p+2}(s)\right\|_{L^1} + 3|\left\langle V_\gamma^{p+1}(s), Z_\gamma(s)\right\rangle| + 3|\left\langle V_\gamma^p(s), H_2(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\right\rangle| \\
&\quad + |\left\langle V_\gamma^{p-1}(s), H_3(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\right\rangle| + 3A\left\|V_\gamma^p(s)\right\|_{L^1} + 3A|\left\langle V_\gamma^{p-1}(s), Z_\gamma(s)\right\rangle|
\end{aligned}$$

Let

$$L_s \stackrel{\text{def}}{=} \|V_\gamma(s)\|_{L^{p+2}}^{p+2}\ , \qquad K_s \stackrel{\text{def}}{=} \left\langle V_\gamma^{p-1}(s), \Delta_\gamma V_\gamma(s)\right\rangle\ .$$

Those terms are the good terms of (3.19), and the idea is now to bound all the other errors $|D(s) - D_1(s)|$ with expression containing L_s and K_s. In the following calculations we assume γ to be small enough such that $|A| \leq \mathfrak{c}_\gamma$. The cost of replacing $D(s)$ with $D_1(s)$

$$
\begin{aligned}
|D(s) - D_1(s)| \leq & \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y) \left|V_\gamma^{p-1}(y,s) - V_\gamma^{p-1}(x,s)\right| \\
& \times \left| \left(X_\gamma^3(y,s) - X_\gamma^3(x,s)\right) - 3(\mathfrak{c}_\gamma + A)\left(X_\gamma(y,s) - X_\gamma(x,s)\right) \right| \\
\leq & \; 3 \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y) \left|V_\gamma^{p-1}(y,s) - V_\gamma^{p-1}(x,s)\right| \\
& \times \left(\left|V_\gamma(y,s) - V_\gamma(x,s)\right| + |Z_\gamma(y,s) - Z_\gamma(x,s)| \right) \left(2\mathfrak{c}_\gamma + X_\gamma^2(x,s)\right) \; .
\end{aligned}
$$

Denote with

$$
\begin{aligned}
D_2 = & \; 3 \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y) \left|V_\gamma^{p-1}(y,s) - V_\gamma^{p-1}(x,s)\right| \\
& \qquad \times |V_\gamma(y,s) - V_\gamma(x,s)| \left(2\mathfrak{c}_\gamma + X_\gamma^2(x,s)\right) \\
D_3 = & \; 3 \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y) \left|V_\gamma^{p-1}(y,s) - V_\gamma^{p-1}(x,s)\right| \\
& \qquad \times |Z_\gamma(y,s) - Z_\gamma(x,s)| \left(2\mathfrak{c}_\gamma + X_\gamma^2(x,s)\right) \; .
\end{aligned}
$$

We will now bound D_3 with a small multiple of L_s and K_s plus an error in (3.18), the term D_2 can be bounded in a similar way.
By $a^n - b^n = (a-b)(a^{n-1} + \cdots + b^{n-1})$ and the generalized Young inequality

$$
\begin{aligned}
a^{p-1} - b^{p-1} = & \; (a-b)(a^{p-2} + \cdots + b^{p-2}) \\
& \leq |a^{p-1} - b^{p-1}| \frac{|a-b|}{2\lambda} + (|a|^{p-1} + |b|^{p-1})\lambda 2^{p-2}
\end{aligned}
$$

Therefore, applying the previous inequality to each summands of D_3 and choosing $\lambda = c_1^{-1}(\gamma^{-1}\epsilon)^2 \, |Z_\gamma(y,s) - Z_\gamma(x,s)| \left(2\mathfrak{c}_\gamma + X_\gamma^2(x,s)\right)$ we have that

$$
\begin{aligned}
D_3 \leq & \; c_1 K_s + C c_1^{-1}(\epsilon^2\gamma^{-2}) \sum_{x\in\Lambda_\varepsilon} \epsilon^2 |V_\gamma^{p-1}(x,s)||Z_\gamma(x,s)| \left(\mathfrak{c}_\gamma + X_\gamma^2(x,s)\right) \\
& \leq c_1 K_s + c_1 L_s + C c_1^{-1} \left\| \epsilon^2\gamma^{-2} |Z_\gamma(s)| \left(\mathfrak{c}_\gamma + X_\gamma^2(s)\right) \right\|_{L^{\frac{p-2}{3}}}^{(p-2)/3} \qquad (3.22)
\end{aligned}
$$

where $c_1 > 0$ can be chosen to be for instance $c_1 = 1/8$. The last term will be part of the error (3.18). Recall that $\epsilon = \gamma^2$ and the last term of (3.22) is bounded in expectation using

Proposition 3.3.1, Lemma B.0.5 and [MW17a, Proposition 5.4]

$$\mu_\gamma\left[\left\|\epsilon^2\gamma^{-2}|Z_\gamma(s)|\left(\mathfrak{c}_\gamma+X^2_\gamma(s)\right)\right\|^{(p-2)/3}_{L^{\frac{p-2}{3}}}\right]\le$$
$$\mu_\gamma\left[\|Z_\gamma(s)\|^{2(p-2)/3}_{\mathcal{C}^{-\nu}(\mathbb{T}^2)}\right]^{1/2}\left((\gamma^2\mathfrak{c}_\gamma)^{\frac{2(p-2)}{3}}+\mu_\gamma\left[\|\gamma X_\gamma(s)\|^{2(p-2)/3}_{L^{2(p-2)/3}}\right]\right)^{1/2}$$
$$\le C(T)\gamma^{\frac{p-2}{6}-2\nu\frac{(p-2)}{3}} \qquad (3.23)$$

which is negligible if ν is small enough and $p>2$. It is immediate to generalize (3.23) to any power, as in (3.18).
We will then bound the term $B(t)$ in (3.20) with Proposition 3.3.1. Using Young's inequality we have that

$$\begin{aligned}B(s)&\le C\gamma^2\left\langle|V^{p-1}_\gamma|(s),\mathfrak{c}_\gamma+\left|Z^2_\gamma(s)+2V_\gamma(s)Z_\gamma(s)+V^2_\gamma(s)\right||X_\gamma|^3(s)\right\rangle\\
&\le\frac{1}{24}\|V_\gamma(s)\|^{p+2}_{L^{p+2}}+C\mathfrak{c}_\gamma^{\frac{p+2}{3}}\left\|(\gamma^{\frac{2}{3}}X_\gamma(s))^{p+2}\right\|_{L^1}\\
&+C\left\|Z_\gamma^{\frac{2p+4}{3}}(\gamma^2X^3_\gamma(s))^{\frac{p+2}{3}}\right\|_{L^1}+\left\|Z_\gamma^{\frac{p+2}{2}}(\gamma^2X^3_\gamma(s))^{\frac{p+2}{2}}\right\|_{L^1}+\left\|(\gamma^2X^3_\gamma(s))^{p+2}\right\|_{L^1}\ .\end{aligned}$$

The constant $1/24$ has been arbitrarily chosen in order to control $B(s)$ with a small multiple of L_s plus a quantity that will be part of the error in (3.18) and can be bounded in expectation, as we did in (3.23), by $C(T)\gamma^{\frac{p+2}{6}-2\nu\frac{2p+4}{3}}$, which is negligible for ν small enough.

We are now in the same setting of [TW16, Eq. 3.13], namely the discrete process V_γ satisfies

$$\begin{aligned}&\|V_\gamma(t)\|^p_{L^p}-p\int_0^t\frac{5}{6}K_s+\frac{5}{24}L_s\,ds \qquad (3.24)\\
&\le\left\|V^0_\gamma\right\|^p_{L^p}+\frac{p}{3}\int_0^t\sum_{j=0}^2\binom{3}{j}\left\langle V_\gamma^{p-1+j}(\cdot,s),H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\right\rangle\,ds\\
&+A\int_0^t|\left\langle V_\gamma^{p-1}(\cdot,s),(V_\gamma(\cdot,s)+Z_\gamma(\cdot,s))\right\rangle|\,ds+\int_0^t\mathsf{Err}(s)ds\,,\end{aligned}$$

where $\mu_\gamma[|\mathsf{Err}(s)|^q]^{\frac{1}{q}}\le C(T,p,q)\gamma^{\frac{p-2}{6}-2\nu\frac{(p-2)}{3}}$ for any positive q.
We will now show that, for ν small enough and $j=0,1,2$, there exist $\lambda_{j,1},\lambda_{j,1}>0$

$$\begin{aligned}&\left\langle V_\gamma^{p-1+j},H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\right\rangle\\
&\lesssim\left(L_s^{\frac{p-1+j}{p+2}-\nu\frac{p}{p+2}}K_s^\nu+L_s^{\frac{p-1+j}{p+2}}\right)\|H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\|_{\mathcal{C}^{-\nu}(\Lambda_\varepsilon)}\\
&\le\frac{1}{7}K_s+\frac{1}{30}L_s+C\sum_{i=1,2}\|H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\|^{\lambda_i}_{\mathcal{C}^{-\nu}(\Lambda_\varepsilon)} \qquad (3.25)\end{aligned}$$

where the last line follows from the Young inequality for ν sufficiently small. In a similar way

$$A|\left\langle V_\gamma^{p-1}(\cdot,s),(V_\gamma(\cdot,s)+Z_\gamma(\cdot,s))\right\rangle| \\ \leq \frac{1}{7}K_s+\frac{1}{30}L_s+C(A)\left(1+\|Z_\gamma(\cdot,s)\|_{\mathcal{C}^{-\nu}(\Lambda_\varepsilon)}^{\lambda_i}\right) \quad (3.26)$$

Recall that all the norms appearing the proof so far are norms on the discrete lattice. The same proof of [TW16, Proposition 3.7] can be used to prove (3.25) and (3.26), provided the same inequalities hold in the discrete setting.
We are going to prove (3.25), (3.26) being essentially the same. Using the duality for discrete Besov spaces proved in Prop. A.0.4

$$\left\langle V_\gamma^{p-1+j}(s),H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\right\rangle_{\Lambda_\varepsilon}\leq\left\|V_\gamma^{p-1+j}(s)\right\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)}\|H_{3-j}(Z_\gamma(\cdot,s),\mathfrak{c}_\gamma)\|_{\mathcal{C}^{-\nu}(\Lambda_\varepsilon)}\,.$$

We then control $\left\|V_\gamma^{p-1+j}(s)\right\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)}$ with Lemma A.0.11. From (A.10) applied to $f(x)=V_\gamma^{p-1+j}(x,s)$

$$\|f\|_{\mathcal{B}_{1,1}^\nu(\Lambda_\varepsilon)}\lesssim\|f\|_{L^1(\Lambda_\varepsilon)}^{1-2\nu}\left(\sum_{x,y\in\Lambda_\varepsilon}\epsilon^4K_\gamma(x-y)\epsilon^{-1}\gamma|f(x)-f(y)|\right)^{2\nu}+\|f\|_{L^1(\Lambda_\varepsilon)}\,.$$

We will now estimate the term inside the brackets. For p even and $j\in\mathbb{N}$, we have

$$|a^{p-1+j}-b^{p-1+j}|^{\frac{p-1}{p-1+j}}\leq|a^{p-1}-b^{p-1}|$$

the above equation follows easily from the Minkowski inequality if one assumes a and b to have the same sign. If the a and b have different signs, the inequality follows by the fact that p is an even integer and hence the right-hand-side is equal to $|a|^{p-1}+|b|^{p-1}$. Therefore from the generalized Young inequality for $\lambda>0$

$$|a^{p-1+j}-b^{p-1+j}|\leq|a^{p-1}-b^{p-1}|^{\frac{p-1+j}{p-1}} \\ \leq\lambda|a^{p-1}-b^{p-1}||a-b|+\frac{C}{\lambda}(|a|^{p-2+2j}+|b|^{p-2+2j})\,,$$

we have for every $\lambda>0$

$$\sum_{x,y\in\Lambda_\varepsilon}\epsilon^4K_\gamma(x-y)\epsilon^{-1}\gamma|V_\gamma^{p-1+j}(x,s)-V_\gamma^{p-1+j}(y,s)| \\ \lesssim\lambda\left\langle V_\gamma^{p-1}(s),\Delta_\gamma V_\gamma(s)\right\rangle+\frac{1}{\lambda}\left\|V_\gamma^{p+2+2j}(s)\right\|_{L^1}$$

and optimizing in λ we get (3.25). Finally we can gather together (3.24) and (3.25) to conclude the proof. $\square$

We remark that the right-hand-side of (3.25) is slightly different from [TW16] since we have to use Δ_γ the discrete (long range) Laplacian, which is a good approximation of the continuous Laplacian only on low frequencies. We now turn to the proof of Theorem 3.2.1, which follows the lines of [TW16, Cor. 3.10].

Proof of Theorem 3.2.1. By the monotonicity of L^q norms it is sufficient to prove the statement of the theorem for q large enough. In the following proof C will denote a constant possibly changing from line to line. The Gibbs measure μ_γ is an invariant measure for the Glauber dynamic. For all $T \geq 0$

$$\mu_\gamma \left[\|X_\gamma\|^q_{\mathcal{C}^{-\nu}} \right] = \frac{2}{T} \int_{T/2}^{T} \mathbb{E}_{\mu_\gamma} \left[\|X_\gamma(\cdot, s)\|^q_{\mathcal{C}^{-\nu}} \right] ds \, . \tag{3.27}$$

From the definition of V_γ we can write

$$\|X_\gamma(\cdot, s)\|_{\mathcal{C}^{-\nu}} \leq \|Z_\gamma(\cdot, s)\|_{\mathcal{C}^{-\nu}} + \|V_\gamma(\cdot, s)\|_{\mathcal{C}^{-\nu}}$$

By (3.12) proven in [MW17a, Prop. 5.4], for all $j \geq 1$ we have that

$$\mathbb{E}_{\mu_\gamma} \left[\sup_{s \in [T/4, T]} \left\| Z_\gamma^{:j:}(\cdot, s) \right\|^q_{\mathcal{C}^{-\nu}} \right] < C(T, q, j) \tag{3.28}$$

where the proportionality constant may depend on T and q. From the definition of the discrete Besov norm it follows that (3.28) holds true also when we replace the continuous discrete Besov norm with the discrete one. By Proposition A.0.3 and Lemma A.0.1, for any $q > d/\nu$ and $\kappa > 0$

$$\begin{aligned} &\|V_\gamma(s)\|_{\mathcal{C}^{-\nu}} \leq \|\mathsf{Ext} V_\gamma(s)\|_{L^q(\mathbb{T}^2)} \\ &\lesssim \|V_\gamma(s)\|_{L^q(\Lambda_\varepsilon)} + \epsilon^{-\kappa} \|V_\gamma(s)\|^{1-\frac{1}{q}}_{L^{2q-2}(\Lambda_\varepsilon)} \left\{ \sum_{\substack{|x-y|=\epsilon \\ x,y \in \Lambda_\varepsilon}} \epsilon^2 (V_\gamma(s,y) - V_\gamma(s,x))^2 \right\}^{\frac{1}{2q}} \end{aligned} \tag{3.29}$$

where the proportionality constant may depends on q and κ. In Proposition 3.3.3, using

(3.28) and (3.18) we obtain that, for $T/4 \le s < t \le T$

$$\begin{aligned}
&\mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,t)\|^q_{L^q(\Lambda_\varepsilon)}\right] + C_1\int_s^t \mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,r)\|^q_{L^q(\Lambda_\varepsilon)}\right]^{\frac{q+2}{q}} dr \\
&\quad + C_1\int_s^t \mathbb{E}_{\mu_\gamma}\left[\left\langle V_\gamma^{q-1}(r),(-\Delta_\gamma)V_\gamma(r)\right\rangle_{\Lambda_\varepsilon}\right] dr \le \mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,s)\|^q_{L^q(\Lambda_\varepsilon)}\right] + C(q,T)
\end{aligned} \tag{3.30}$$

From Lemma A.0.12, applied to $\mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,t)\|^q_{L^q(\Lambda_\varepsilon)}\right]$ we have that there exists $C(q,T)$ such that for all $T/4 \le s \le t \le T$ we have

$$\mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,t)\|^q_{L^q(\Lambda_\varepsilon)}\right] \lesssim C(q,T)\left(|t-s|^{-\frac{q}{2}} \vee 1\right) .$$

Let us choose $s = T/4$ and $t \in [T/2,T]$: from the above inequality we have that

$$\sup_{T/2\le t\le T} \mathbb{E}_{\mu_\gamma}\left[\|V_\gamma(\cdot,t)\|^q_{L^q(\Lambda_\varepsilon)}\right] \le C(q,T) . \tag{3.31}$$

At this point we only need to provide a bound for

$$\sum_{\substack{|x-y|=\epsilon \\ x,y\in\Lambda_\varepsilon}} \epsilon^2 (V_\gamma(y,s) - V_\gamma(x,s))^2 \lesssim \epsilon^2 \sum_{\omega\in\Lambda_N} |\omega|^2 |\hat{V}_\gamma(s)|^2 .$$

By (B.0.1), the operator Δ_γ approximates the discrete Laplacian only for low frequencies $|\omega| \le \gamma^{-1}$

$$|\widehat{\Delta_\gamma V_\gamma}(s)(\omega)| = \gamma^{-2}(1-\hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)| \ge c|\omega|^2|\hat{V}_\gamma(s)(\omega)| .$$

On the other hand, for high frequencies $\gamma^{-1} \le |\omega| \le \gamma^{-2}$, we have

$$|\widehat{\Delta_\gamma V_\gamma}(s)(\omega)| \ge \gamma^{-2}(1-\hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)| \ge \gamma^{-2}|\hat{V}_\gamma(s)(\omega)| ,$$

hence for all $\omega \in \Lambda_N$,

$$|\omega|^2|\hat{V}_\gamma(s)|^2 \le \gamma^{-2}(|\omega|^2 \wedge \gamma^{-2})|\hat{V}_\gamma(s)(\omega)|^2 \le \gamma^{-4}(1-\hat{K}_\gamma(\omega))|\hat{V}_\gamma(s)(\omega)|^2$$

and therefore

$$\sum_{\omega\in\Lambda_N} |\omega|^2|\hat{V}_\gamma(s)|^2 \le \gamma^{-2}\left\langle V_\gamma(s),(-\Delta_\gamma)V_\gamma(s)\right\rangle_{\Lambda_\varepsilon} .$$

Equation (3.30) holds for any positive even integers $q \ge 2$ and $T/4 \le s \le t \le T$, if we choose $s = T/2$, $t = T$ and $q = 2$ we conclude that

$$\int_{T/2}^{T} \sum_{|x-y|=\epsilon} \epsilon^2 (V_\gamma(y,s) - V_\gamma(x,s))^2 ds \\ \leq \epsilon^2 \gamma^{-2} \int_{T/2}^{T} \mathbb{E}_{\mu_\gamma} \left[\langle V_\gamma(r), (-\Delta_\gamma) V_\gamma(r) \rangle_{\Lambda_\varepsilon} \right] dr \leq C(T) \ . \quad (3.32)$$

It is sufficient now to control the right-hand-side of (3.27) with (3.29). Again we can choose $s = T/2$, $t = T$ in (3.30): By (3.28), (3.31) and (3.32)

$$\begin{aligned}
&\mu_\gamma \big[\, \|X_\gamma\|^q_{\mathcal{C}^{-\nu}} \, \big] \leq \frac{2}{T} \int_{T/2}^{T} \mathbb{E}^\gamma_{\beta,0} \big[\, \|Z_\gamma(s)\|^q_{\mathcal{C}^{-\nu}} \, \big] + \mathbb{E}^\gamma_{\beta,0} \big[\, \|V_\gamma(s)\|^q_{\mathcal{C}^{-\nu}} \, \big] ds \\
&\leq C(T,q,\kappa) \int_{T/2}^{T} \mathbb{E}^\gamma_{\beta,0} \big[\, \|Z_\gamma(s)\|^q_{\mathcal{C}^{-\nu}} \, \big] + \mathbb{E}^\gamma_{\beta,0} \big[\, \|V_\gamma(s)\|^q_{L^q(\Lambda_\varepsilon)} \, \big] ds \\
&+ C(T,q,\kappa) \, \mathbb{E}^\gamma_{\beta,0} \int_{T/2}^{T} \|V_\gamma(s)\|^{q-1}_{L^{2q-1}(\Lambda_\varepsilon)} \, \epsilon^{-q\kappa} \Bigg\{ \sum_{\substack{|x-y|=\epsilon \\ x,y \in \Lambda_\varepsilon}} \epsilon^2 (V_\gamma(s,y) - V_\gamma(s,x))^2 \Bigg\}^{\frac{1}{2}} ds \\
&\leq C(T,q,\kappa) \left(1 + \epsilon^{-q\kappa} \epsilon \gamma^{-1} \left\{ \int_{T/2}^{T} \mathbb{E}^\gamma_{\beta,0} \left[\langle V_\gamma(r), (-\Delta_\gamma) V_\gamma(r) \rangle_{\Lambda_\varepsilon} \right] dr \right\}^{1/2} \right) \ ,
\end{aligned}$$

where in the last line we applied the Cauchy-Schwarz inequality

$$\mathbb{E}^\gamma_{\beta,0} \left[\|V_\gamma(s)\|^{q-1}_{L^{2q-1}(\Lambda_\varepsilon)} \Bigg\{ \sum_{\substack{|x-y|=\epsilon \\ x,y \in \Lambda_\varepsilon}} \epsilon^2 (V_\gamma(s,y) - V_\gamma(s,x))^2 \Bigg\}^{\frac{1}{2}} \right] \\ \leq \left\{ \mathbb{E}^\gamma_{\beta,0} \big[\, \|V_\gamma(s)\|^{2q-2}_{L^{2q-1}(\Lambda_\varepsilon)} \, \big] \epsilon^2 \gamma^{-2} \mathbb{E}^\gamma_{\beta,0} \big[\, \langle V_\gamma(r), (-\Delta_\gamma) V_\gamma(r) \rangle_{\Lambda_\varepsilon} \, \big] \right\}^{\frac{1}{2}} .$$

The claim follows by choosing $\kappa > 0$ small enough. □

Chapter 4

Fluctuation of Kawasaki dynamic in a 1-d Ising-Kac model

The chapter that follows can be considered more a collection of techniques, rather than a properly structured chapter. Indeed we do not have, in this moment, a complete proof of the main result Theorem 4.2.7. We will show however a conditional result based on the validity of the replacement lemma given in Conjecture 4.2.6. We believe the proof of the conditional result to be rigorous but we wish to inform the reader of the lack of detail in some of the forthcoming propositions.

4.1 Introduction

We are considering the Kawasaki dynamic on a Ising model on the one dimensional discrete lattice $\Lambda_N = \{-N+1, \dots, N\}$ with periodic boundary conditions and Hamiltonian

$$\mathcal{H}_\gamma^{\Lambda_N}(\sigma) = \frac{\beta}{2} \sum_{x,y \in \Lambda_N} \kappa_\gamma(x,y)\sigma_x\sigma_y \tag{4.1}$$

Here $\sigma \in \Sigma_N := \{-1,1\}^{\Lambda_N}$ is the spin configuration, β is the inverse temperature and κ_γ, for $\gamma > 0$, is the Kac potential defined in (1.5). Recall furthermore the definition of the local magnetization h_γ in Section 1.2.1.

Denote with $\mu_{\gamma,\beta,b}^{\Lambda_N}$ the Grand Canonical Gibbs measure associated to the above Hamiltonian on the lattice Λ_N

$$\mu_{\gamma,\beta,b}^{\Lambda_N}(\sigma) \stackrel{\text{def}}{=} \left(\mathcal{Z}_{\gamma,\beta,b}^{\Lambda_N}\right)^{-1} \exp\left\{\mathcal{H}_\gamma^{\Lambda_N}(\sigma) + \beta b \sum_{x\in\Lambda_N} \sigma_x\right\} \tag{4.2}$$

with partition function $\mathcal{Z}^{\Lambda_N}_{\gamma,\beta,b}$. In this chapter, for brevity the domain Λ_N will be often omitted form the notation of the Hamiltonian or the Gibbs measure, and we will use the notation μ_γ when the external magnetization $b = 0$.
For a configuration $\sigma \in \Sigma_N$, we will denote with $\sigma^{\{x,y\}}$ the configuration

$$\sigma^{\{x,y\}}_z = \begin{cases} \sigma_x & \text{if } z = y \\ \sigma_y & \text{if } z = x \\ \sigma_z & \text{otherwise} \end{cases}$$

where the values at sites x and $y \in \Lambda_N$ have been exchanged.
We are interested in the Kawasaki dynamic, a conservative, spin exchange, stochastic dynamic with state space Σ_N, according to which two nearest neighbors sites $x \sim y \in \Lambda_N$ exchange their magnetization with rate that depends on the energy difference between the configuration (see [KL99, Lig05, Spo91])

$$c^K_\gamma(x,y,\sigma) = \Phi\left(\mathcal{H}_\gamma(\sigma^{\{x,y\}}) - \mathcal{H}_\gamma(\sigma)\right) .$$

In this chapter we will focus on a precise choice for Φ

$$\Phi(H) = \frac{e^{\frac{H}{2}}}{\cosh(\frac{H}{2})} = 1 + \tanh\left(\frac{H}{2}\right)$$

hence the rate function satisfies

$$c^K_\gamma(x,y,\sigma) = 1 - \frac{\sigma_y - \sigma_x}{2}\tanh\left(\beta[h_\gamma(y) - h_\gamma(x)] + \beta\kappa_\gamma(x,y)(\sigma_y - \sigma_x)\right) . \tag{4.3}$$

The generator associated to the Kawasaki dynamic applied to functions $g : \Sigma_N \to \mathbb{R}$ is the written as

$$\mathscr{L}^K g(\sigma) = \sum_{x\sim y\in\Lambda_N} c^K_\gamma(x,y,\sigma)\left(g(\sigma^{x,y}) - g(\sigma)\right) \tag{4.4}$$

where in the sum $x \sim y \in \Lambda_N$ every couple of neighboring sites is counted once. It is easy to see that the rates (4.3) satisfy the following conditions

- Coercivity and Boundedness: there are $c, C > 0$ such that, uniformly in γ

$$c \leq c^K_\gamma(x,y,\sigma) \leq C \qquad \forall x,y \in \Lambda_N\ \forall\sigma \in \Sigma_N . \tag{CB}$$

- Finite range: for fixed $\gamma > 0$ and $|x - y| = 1$, the rate $c^K_\gamma(x,y,\sigma)$ depends only on the values of the spins inside a ball centered in x of radius $4\gamma^{-1}$.

- Translation invariance: for $z \in \Lambda_N$, let τ_z be the translation satisfying $\tau_z \sigma_x = \sigma_{x+z}$ for all $x \in \Lambda_N$, then

$$c_\gamma^K(x-z, y-z, \tau_z \sigma) = c_\gamma^K(x, y, \sigma) .$$

- Reversibility: for any $b \in \mathbb{R}$, with respect to $\mu_{\gamma,b}$, the rates satisfy the detailed balance conditions

$$c_\gamma^K(x, y, \sigma)\mu_{\gamma,b}(\sigma) = c_\gamma^K(x, y, \sigma^{\{x,y\}})\mu_{\gamma,b}(\sigma^{\{x,y\}}) . \tag{DB}$$

A major difficulty for the Kawasaki dynamic (and for many dynamics with conservation law) is the fact that it doesn't satisfy the so-called *gradient condition*. In order to formulate it, will introduce what in the context of lattice gasses is the current

$$w_{x,y}(\sigma) = c_\gamma^K(x, y, \sigma)(\sigma_x - \sigma_y) \tag{4.5}$$

- Gradient condition: there exist a local function $\Psi : \Sigma_N \to \mathbb{R}$ such that

$$w_{x,y}(\sigma) = \tau_x \Psi(\sigma) - \tau_y \Psi(\sigma)$$

  here we used the notation $\tau_x \Phi(\sigma) := \Phi(\tau_x \sigma)$, where $\tau_x \sigma$ has been defined in one of the points above.

The gradient condition yields important simplifications when considering hydrodynamic limits of the local magnetization, because it allows to write the microscopic current as a gradient of a function of the field.

For non-gradient systems, however, it is possible to recover the hydrodynamic limit decomposing the microscopic current as the sum of a gradient part and a fluctuating part that is vanishing in the limit. For the Kawasaki dynamic described above, in [VY97] it is proven that the density field $\eta_x = 2^{-1}\{\sigma_x + 1\}$ (which codifies the absence or the presence of a particle) converges weakly, under a diffusive scaling to the solution of the following nonlinear equation

$$\begin{cases} \partial_t \rho(t,x) & = \partial_x \left(D_\beta \left(\rho(t,x) \right) \partial_x \rho(t,x) \right) \\ \rho(0,x) & = \rho_0(x) \end{cases}$$

where the diffusion matrix $D_\beta(r)$ is defined via a variational formula (see [VY97, Eq. 2.22]). The nongradient method requires usually some information about the invariant measure (mixing conditions, spectral gap in finite domains) which are satisfied in case β is sufficiently close to 0, hence not in the regime studied in this thesis.

In our case the Hamiltonian depend on the interaction κ_γ with range γ^{-1} that is going to infinity as we take the hydrodynamic limit $N \to \infty$. In [Gia91, LOP91] the authors studied a model (where $\gamma^{-1} \ll N$) in one dimension generalizing a technique based on a propagation of chaos for product measures used in [DMPS89] for the weakly asymmetric simple exclusion process. In the above papers the diffusion matrix is shown to be given by

$$D_\beta(r) = \frac{1}{2} - \beta\, 2r(1-r)$$

if the initial condition $\rho_0 \in [0,1]$ lies in a connected component of the subset $D(r) > 0$. A propagation of chaos result ensures that the coercivity of the diffusion matrix is preserved during the time and this guarantees that the process rescales diffusively. Our work is motivated by the considerations about the Kawasaki dynamic in [GLP99], where β is chosen close to its critical value 1 and the initial density close to $\frac{1}{2}$. Around a critical point, the diffusion matrix vanishes (see [SY95] for rigorous estimates in case of nearest neighbor interaction) and one has to investigate the spin system/particle system under a different scale.

In [GL97], the case $\gamma^{-1} \simeq N$ has been treated and the local density field is shown to converge to the solution of an integrodifferential equation that depends on the shape on the interaction $\kappa_\gamma(x,y) \simeq \gamma^d \mathfrak{K}(\gamma|x-y|)$, that is given, in the language of magnetic fields,

$$\partial_t m = \nabla \cdot \Big(\frac{\beta}{2}(1-m^2)\nabla \frac{\delta \mathcal{F}^{LP}}{\delta m}(m)\Big) \tag{4.6}$$

where $\mathcal{F}^{LP}$ is the Lebowitz-Penrose functional (see [LP66] or [Pre09, Chapter 4]) given by

$$\mathcal{F}^{LP}(m) = \int_{\mathbb{T}^d} F^{MF}(m)\, dx + \frac{1}{4}\int_{\mathbb{T}^d \times \mathbb{T}^d} \mathfrak{K}(|x-y|)(m(x)-m(y))^2\, dxdy$$

and F^{MF} is the mean field free energy

$$F^{MF}(m) = \frac{1}{\beta}\left(\frac{1+m}{2}\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)\right) - \frac{m^2}{2}\,. \tag{4.7}$$

The free energy (4.7) encodes information about the thermodynamic of the model: as long as $\beta \le 1$ (4.7) is a concave function and this brings numerous consequences about the existence of a unique Gibbs measure in infinite volume. On the other hand, if $\beta > 1$, (4.7) has two minima, and this is a symptom of the phase transition. The main consequence of working at critical temperature is the particular rescaling of the model: in the literature on hydrodynamic limits it is often natural to work in a diffusive regime since $D(r) > 0$ forces

to rescale the time variable twice as much as the space variable. However, if r is fluctuating around $\frac{1}{2}$ and $D(1/2) = 0$, the rescaling of the time variable is given by the next order expansion of $D(r) \sim 2(r)^2$. A more careful analysis reveals also the presence of a fourth order positive operator that improves the regularity of the solution and guarantees the well posedness of the limiting SPDE.
A connection between the hydrodynamic limit of the Kawasaki dynamic and the Cahn-Hilliard equation is obtained from (4.6) for small $m \to 0$, writing the Cahn-Hilliard equation as (4.6) where the functional $\mathcal{F}^{LP}$ is then approximated by

$$\mathcal{F}^{CH}(m) = \int_{\mathbb{T}^d} F^{(CH)}(m(x))\, dx + \int_{\mathbb{T}^d} |\nabla m(x)|^2 \, dxdy$$

where $F^{CH}(m)$ is a double well functional of the form

$$F^{CH}(m) = -\frac{1}{2}m^2 + \frac{1}{12}m^4 \, .$$

In the present work we study the fluctuation of the magnetization field of the Ising-Kac model under the Kawasaki dynamic in one dimension around the critical temperature, and we prove that, conditionally on Conjecture 4.2.6, the fluctuation field converges to the solution of the stochastic Cahn-Hilliard equation

$$\begin{cases} dX(t) = & -\Delta\left(\Delta X(t) - \frac{1}{3}X^3(t) + AX(t)\right)dt + \sqrt{2}\zeta \\ X(0) = & X^0 \end{cases} \tag{4.8}$$

where $A \in \mathbb{R}$ and ζ is a (degenerate) Gaussian space-time noise on $[0,T] \times \mathbb{T}$ with covariance

$$\mathbb{E}\left[\zeta(\phi)\zeta(\psi)\right] = \int_{[0,T]\times\mathbb{T}} \phi(t,x)(-\Delta)\psi(t,x)\, dt \, .$$

We want to remark that the noise in (4.8) is white in time but coloured in space and does not act on the zero-th Fourier mode. This is a consequence of the fact that the microscopic dynamic is conservative.

4.2 Notations and main result

As in the previous chapters, we will think of Λ_N as a discretization of the 1-dimensional torus $\mathbb{T} := [-1,1]$, so define $\epsilon = N^{-1}$ and $\Lambda_\varepsilon = [-1,1] \cap (\epsilon\mathbb{Z}) \subseteq \mathbb{T}$. With respect to this coordinates define

$$\nabla_\epsilon f(z) = \epsilon^{-1}\left(f(z+\epsilon) - f(z)\right)$$
$$\Delta_\epsilon f(z) = \epsilon^{-2}\left(f(z+\epsilon) - 2f(z) + f(z-\epsilon)\right)$$

For convenience we will also use the discrete gradient and Laplacian over functions in the lattice $g : \Lambda_N \to \mathbb{R}$:

$$\nabla_N^+ g(x) = g\,(x+1) - g\,(x)$$
$$\Delta_N g(x) = g\,(x+1) - 2g\,(x) + g\,(x-1)$$

for $x \in \Lambda_N$.

Recall the definition of the kernel $\mathfrak{K} : B(0,3) \to [0,1]$ a positive, compactly supported, isotropic function which is twice differentiable and satisfies

$$\int_{\mathbb{R}} \mathfrak{K}(z)dz = 1\ .$$

In order to simplify some of the calculations, we assume moreover the kernel $\mathfrak{K}$ to be flat in a small neighbourhood of the origin

Assumption 4.2.1 *There exists* $\mathbf{a} > 0$ *such that the kernel* $\mathfrak{K}$ *is flat in a ball of radius* a *around the origin*

$$\mathfrak{K}(z) = \mathfrak{K}(0) \qquad \forall z \in B_0(\mathbf{a}) \tag{FLAT}$$

The above assumption doesn't seem to be strictly necessary, but it will be convenient for the calculation of the spectral gap of the dynamic in small blocks. From $\mathfrak{K}$, we recall the Kac interaction used in (4.1)

$$\kappa_\gamma(x) \stackrel{\text{def}}{=} \frac{\gamma\,\mathfrak{K}(\gamma x)}{\sum_{x\in\Lambda_N\setminus\{0\}} \gamma\,\mathfrak{K}(\gamma x)} \qquad x \in \Lambda_N \setminus \{0\} \tag{4.9}$$

and $\kappa_\gamma(0) = 0$. The definition of κ_γ is consistent with the one provided in the previous chapters regarding Glauber type dynamics. For the Kawasaki dynamic it turns out that is more convenient to have also a modified version of κ_γ which is flat at the origin (roughly speaking if in Glauber type dynamics we want the kernel to be zero at the origin, for Kawasaki dynamics it is more convenient to ask the gradient of the kernel to be zero at the origin). We therefore define the kernel

$$\tilde{\kappa}_\gamma(x) \stackrel{\text{def}}{=} \begin{cases} \kappa_\gamma(x) & \text{for } x \in \Lambda_N \setminus \{0\} \\ \kappa_\gamma(1) & \text{for } x = 0 \end{cases} .$$

Here we used $\kappa_\gamma(1)$ because $\kappa_\gamma(0) = 0$ according to our definition, and in this way we preserve the flatness of the kernel. We want to remark that $\tilde{\kappa}_\gamma$ is a kernel with (discrete) integral $1 + \kappa_\gamma(1)$.
Define

$$h_\gamma(x,t) = \sum_{z \in \Lambda_N} \kappa_\gamma(x-z)\sigma_z(t) \tag{4.10}$$

$$\tilde{h}_\gamma(x,t) = \sum_{z \in \Lambda_N} \tilde{\kappa}_\gamma(x-z)\sigma_z(t) = h_\gamma(x,t) + \kappa_\gamma(1)\sigma_x(t) . \tag{4.11}$$

The kernels κ_γ and $\tilde{\kappa}_\gamma$ differs only from their value in the origin, they are interchangeable for the purpose of defining the Hamiltonian (4.1). The quantity (4.10) is the same local magnetization studied in [MW17a] and [BPRS93] in the case of the Glauber dynamic. In the case of the Kawasaki process, it turns out, see also [Pen91], that a more natural quantity to consider is $\tilde{h}_\gamma$.

Following the approach of [BPRS93] and [MW17a], we are interested in the evolution of the local magnetization $h_\gamma(z,t)$ in (4.10) given by

$$h_\gamma(z,t) = h_\gamma(z,0) + \int_0^t \mathscr{L}^K h_\gamma(z,s^-)ds + m_\gamma(z,t) \tag{4.12}$$

where $\mathscr{L}^K$ is the generator of the Kawasaki dynamic, defined in (4.4) and $m_\gamma(z,t)$ is a martingale having quadratic variation

$$\begin{aligned} &\langle m_\gamma(z_1,\cdot), m_\gamma(z_2,\cdot)\rangle_t \\ &= 2\int_0^t \sum_{x \in \Lambda_N} c_\gamma^K(x,x+1,\sigma(s^-))(1-\sigma_x\sigma_{x+1}(s^-))\nabla_N^+\kappa_\gamma(x-z_1)\nabla_N^+\kappa_\gamma(x-z_2)\, ds . \end{aligned}$$

For a function $\varphi : \Lambda_N \to \mathbb{R}$,

$$\begin{aligned} \left\langle \varphi, \mathscr{L}^K h_\gamma(s)\right\rangle_{\Lambda_N} &= \langle \Delta_N\varphi, h_\gamma(s)\rangle_{\Lambda_N} \\ &\quad + \epsilon \sum_{x \in \Lambda_N} \nabla_N^+ (\kappa_\gamma * \varphi)(x)(1-\sigma_x(s)\sigma_{x+1}(s)) \tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,s)\right) \end{aligned} \tag{4.13}$$

where $\kappa_\gamma * \varphi(x) = \sum_{z\in\Lambda_N} \kappa_\gamma(z-x)\varphi(z)$. As we already mentioned, the Kawasaki dynamic doesn't satisfy the gradient condition, and it is not possible to perform a second summation by parts at the discrete level. However the following considerations show that the gradient condition might be recovered macroscopically, thanks to local ergodic proprieties of the dynamic, if we take advantage of the form of the invariant measure (4.2).
Adding and subtracting $\Delta_N \kappa_\gamma * \tanh(\beta h_\gamma(\cdot, s))(z)$, we can divide the generator into

$$\mathscr{L}^K h_\gamma(z,s) = \mathscr{L}_1^K h_\gamma(z,s) + \mathscr{L}_2^K h_\gamma(z,s)$$

where

$$\mathscr{L}_1^K h_\gamma(z,s) = \Delta_N h_\gamma(z,s) - \kappa_\gamma * \Delta_N \tanh(\beta h_\gamma(\cdot, s))(z) \tag{4.14}$$

$$\begin{aligned} \mathscr{L}_2^K h_\gamma(z,s) &= \kappa_\gamma * \Delta_N \tanh(\beta h_\gamma(\cdot, s))(z) \\ &\quad + \sum_{x\in\Lambda_N} \nabla_N^+ \kappa_\gamma(x-z)(1-\sigma_x(s)\sigma_{x+1}(s)) \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma(x,t)\right) . \end{aligned} \tag{4.15}$$

This subdivision highlights what we will consider the *gradient* part and the *nongradient* part of the generator. From the form of the (Grand Canonical) Gibbs measure $\mu_{\gamma,b}$, recall the following formula

$$\mu_{\gamma,b}[\sigma_x f(\sigma)] = \mu_{\gamma,b}[\tanh(\beta h_\gamma(x) + b) f(\sigma)] , \tag{4.16}$$

valid for any local function f whose support does not contain $x \in \Lambda_N$.
This suggests that, under $\mu_{\gamma,b}$, it should be possible to replace the microscopic quantity $\sigma_x\sigma_{x+1}$ in (4.13) with a function of its local average h_γ and recover the gradient condition using the formula for the sum of angles for hyperbolic tangents.

$$\begin{aligned} &(1-\sigma_x(s)\sigma_{x+1}(s)) \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma(x,t)\right) \\ &\quad \approx \left(1 - \tanh\left(\beta \tilde{h}_\gamma(x,t) + b\right) \tanh\left(\beta \tilde{h}_\gamma(x+1,t) + b\right)\right) \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma(x,t)\right) \\ &\qquad\qquad = \tanh\left(\beta \tilde{h}_\gamma(x+1,t) + b\right) - \tanh\left(\beta \tilde{h}_\gamma(x,t) + b\right) . \end{aligned}$$

We will now infer the correct scaling for the equilibrium fluctuations, from $\mathscr{L}_1^K h_\gamma(z,s)$, and later provide a motivation towards the fact that $\mathscr{L}_2^K h_\gamma(z,s)$ is negligible for the dynamic. For the rest of the section we are going to assume $b = 0$, but we would like to stress that the decomposition $\mathscr{L} = \mathscr{L}_1^K + \mathscr{L}_2^K$ can accommodate the case of a nonzero external magnetization very nicely, yielding a gradient structure regardless the size of b.

For macroscopic parameters $x \in \Lambda_\varepsilon, t \in [0,T]$ define the fluctuation field as

$$X_\gamma(x,t) \stackrel{\text{def}}{=} \delta^{-1} h_\gamma(\epsilon^{-1}x, \alpha^{-1}t) \tag{4.17}$$

where the parameters δ, ϵ, α are going to be suitably chosen below and represents the scaling of, respectively, the fluctuations, the space and the time.
The kernel κ_γ has sum 1 and support in a macroscopic ball of radius $\epsilon\gamma^{-1}$ hence we can define its macroscopic version as $K_\gamma(z) = \epsilon^{-d}\kappa_\gamma(\epsilon^d z)$ for $z \in \Lambda_\varepsilon$ and the operator $\Delta_\gamma f = \epsilon^{-2}\gamma^2(K_\gamma *_\epsilon f - f)$, defined for macroscopic functions $f : \Lambda_\varepsilon \to \mathbb{R}$. By (4.14) we have

$$\begin{aligned}\mathscr{L}_1^K X_\gamma(x,t) = \epsilon^2\left(\Delta_\epsilon X_\gamma(x,t) - \Delta_\epsilon K_\gamma *_\epsilon X_\gamma(x,t)\right) - \epsilon^2(\beta-1)\Delta_\epsilon K_\gamma *_\epsilon X_\gamma(x,t) \\ + \epsilon^2 \Delta_\epsilon K_\gamma *_\epsilon \left(\delta^{-1}\tanh(\delta\beta X_\gamma(\cdot,t)) - \beta X_\gamma(\cdot,t)\right)(x)\end{aligned}$$

and we can reformulate (4.12) in terms of the macroscopic variable as

$$\begin{aligned}X_\gamma(x,t) = X_\gamma(x,0) + \frac{\epsilon^4}{\alpha\gamma^2}\int_0^t (-\Delta_\gamma)\Delta_\epsilon X_\gamma(x,s^-)\,ds \\ - \frac{\epsilon^2(\beta-1)}{\alpha}\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X_\gamma(x,s^-)\,ds + \frac{\epsilon^2\delta^2}{\alpha}\frac{\beta^3}{3}\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X_\gamma^3(x,s^-)\,ds \\ + \frac{\epsilon^2\delta^2}{\alpha}\int_0^t \Delta_\epsilon K_\gamma *_\epsilon \mathcal{O}(\delta^2 X_\gamma^5)\,ds + \alpha^{-1}\int_0^t \mathscr{L}_2^K X_\gamma(x,s^-)ds \\ + M_\gamma(x,t)\,.\end{aligned} \tag{4.18}$$

At this point we notice some similarities with [MW17a, Eq. 2.13], but also the presence of an extra term compensating the second summation by parts.
The martingale $M_\gamma(x,t)$ has quadratic variation given by

$$\begin{aligned}&\langle M_\gamma(x,\cdot), M_\gamma(y,\cdot)\rangle_t \\ &\quad = \frac{4\epsilon^4}{\alpha\delta^2}\int_0^t \sum_{z\in\Lambda_N} c_\gamma^K(z,z+1,\sigma(\alpha^{-1}s^-))1_{\{\sigma_z(\alpha^{-1}s^-)\neq\sigma_{z+1}(\alpha^{-1}s^-)\}} \\ &\qquad\qquad \times (\nabla_\epsilon K_\gamma(x-\epsilon z))(\nabla_\epsilon K_\gamma(y-\epsilon z))\,ds\end{aligned}$$

hence we have

$$
\begin{aligned}
&\big\langle M_\gamma(x,\cdot), M_\gamma(y,\cdot)\big\rangle_t \\
&= \frac{\epsilon^3(2+\mathcal{O}(\gamma))}{\alpha\delta^2}\int_0^t \sum_{z\in\Lambda_N} \epsilon(1-\sigma_z(s^-)\sigma_{z+1}(s^-))\,\nabla_\epsilon K_\gamma(x-\epsilon z)\,\nabla_\epsilon K_\gamma(y-\epsilon z)ds\ .
\end{aligned}
\tag{4.19}
$$

Using (4.18) and (4.19) we define the scaling parameters in order to satisfy $\frac{\epsilon^4}{\alpha\delta^2}=\frac{\epsilon^2\delta^2}{\alpha}=\frac{\epsilon^3}{\alpha\delta^2}=1$, yielding

$$
\epsilon=\gamma^{\frac{4}{3}} \qquad \alpha=\gamma^{\frac{10}{3}} \qquad \delta=\gamma^{\frac{1}{3}} \qquad \beta=1+A\gamma^{\frac{2}{3}} \tag{4.20}
$$

Remark 4.2.2 The scaling of the space and the fluctuation field (4.20) coincide with the one used in [BPRS93], the scaling of the time is different, as expected, since the limiting operator is of fourth order in our case.

Remark 4.2.3 It is easy to see that for $d\geq 1$ the quadratic variation of the martingale would have been

$$
\langle M_\gamma(\varphi,\cdot), M_\gamma(\varphi,\cdot)\rangle_t \sim 2\frac{\epsilon^{2+d}}{\alpha\delta^2}t\,\|K_\gamma * \nabla_\epsilon\varphi\|^2_{L^2(\mathbb{T}^d)}\ .
$$

The observation implies that for general dimension, the scaling would have to satisfy $\frac{\epsilon^4}{\alpha\delta^2}=\frac{\epsilon^2\delta^2}{\alpha}=\frac{\epsilon^{2+d}}{\alpha\delta^2}=1$ and therefore

$$
\epsilon=\gamma^{\frac{4}{4-d}} \qquad \alpha=\gamma^{2\frac{4+d}{4-d}} \qquad \delta=\gamma^{\frac{d}{4-d}}\ . \tag{4.21}
$$

The above scaling is meaningful up to $d<4$. Indeed the dimension 4 is the critical dimension for the measure Φ^4_d.

In the rest of the chapter we are not going to use the decomposition $\mathscr{L}^K=\mathscr{L}^K_1+\mathscr{L}^K_2$, but a similar one, which suits best the usual 1-block/2-block approach that we are going to use in the proof, and creates more manageable errors at least in dimension 1.
The new decomposition simply consists in the replacement

$$
\sigma_x\sigma_{x+1}\longrightarrow \mathrm{Av}_{i\neq j}\,\kappa_\gamma(x-i)\kappa_\gamma(x+1-j)\sigma_i\sigma_j=(1+\mathfrak{g}_\gamma)h_\gamma(x)h_\gamma(x+1)-\mathfrak{g}_\gamma
$$

where

$$\mathfrak{g}_\gamma = \left(1 - \sum_{x\in\Lambda_N} \kappa_\gamma(x)\kappa_\gamma(x+1)\right)^{-1} \sum_{x\in\Lambda_N} \kappa_\gamma(x)\kappa_\gamma(x+1) = \mathcal{O}(\gamma)\,. \tag{4.22}$$

This allows a second summation by parts in (4.13) if we approximate the hyperbolic tangent up to the third order

$$\begin{aligned}
&(1-\sigma_x\sigma_{x+1})\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,s)\right)\\
&\qquad\approx (1-(1+\mathfrak{g}_\gamma)h_\gamma(x,s)h_\gamma(x+1,s)+\mathfrak{g}_\gamma)\tanh\left(\beta\nabla_N^+ h_\gamma(x,s)\right)\\
&\qquad= (1+\mathfrak{g}_\gamma)\beta\left(1-h_\gamma(x,s)h_\gamma(x+1,s)\right)\nabla_N^+ h_\gamma(x,s)+\mathcal{O}(\gamma^3)
\end{aligned}$$

where we used the fact that $|h_\gamma(x+1,s)-h_\gamma(x,s)| \lesssim \gamma$ deterministically. To see the summation by parts it is sufficient to use the fact that

$$\begin{aligned}
&3h_\gamma(x,s)h_\gamma(x+1,s)\left(h_\gamma(x+1,s)-h_\gamma(x,s)\right)\\
&\qquad= \nabla_N^+ h_\gamma^3(x,s) - \left(h_\gamma(x+1,s)-h_\gamma(x,s)\right)^3 = \nabla_N^+ h_\gamma^3(x,s)+\mathcal{O}(\gamma^3)\,.
\end{aligned}$$

For convenience, define

$$\begin{aligned}
\mathcal{U}_\gamma(x,s) &\stackrel{\text{def}}{=} \delta^{-1}\alpha^{-1}\sum_{z\in\Lambda_N}\nabla_N^+\kappa_\gamma(\epsilon^{-1}x-z)\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(z,s)\right)\\
&\qquad\times\left\{(1+\mathfrak{g}_\gamma)h_\gamma(x,s)h_\gamma(x+1,s)-\mathfrak{g}_\gamma-\sigma_z(\alpha^{-1}s)\sigma_{z+1}(\alpha^{-1}s)\right\}\,.
\end{aligned}\tag{4.23}$$

Therefore, using the scaling defined in (4.20) and the replacement above, we can rewrite (4.18), for all $x\in\Lambda_\varepsilon$ and $s\in\mathbb{R}^+$

$$\begin{aligned}
X_\gamma(x,t) &= X_\gamma^0(x) - \int_0^t \Delta_\gamma\Delta_\epsilon X_\gamma(x,s)\,ds\\
&\quad - A_\gamma\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X_\gamma(x,s)\,ds + \frac{B_\gamma}{3}\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X_\gamma^3(x,s)\,ds + M_\gamma(x,t)\\
&\quad + \int_0^t \mathcal{U}_\gamma(x,s)ds + \int_0^t \mathsf{Err}_2(x,s)ds
\end{aligned}\tag{4.24}$$

where

$$A_\gamma = \delta^{-2}(\beta-1)(1+\mathfrak{g}_\gamma) = A+\mathcal{O}(\gamma^{\frac{1}{3}})\,,\qquad B_\gamma = \beta(1+\mathfrak{g}_\gamma) = 1+\mathcal{O}(\gamma^{\frac{1}{3}})$$

and $\forall x \in \Lambda_\varepsilon$ and $s > 0$

$$|\mathsf{Err}_2(x,s)| \lesssim \delta^{-1}\alpha^{-1} \sum_{x\in\Lambda_N} |\kappa_\gamma(x+1) - \kappa_\gamma(x)|\gamma^3 \lesssim \gamma^{\frac{1}{3}}\,.$$

In order to work with (4.24) we apply an idea of [DPD03], already used successfully in [MW17a] and Chapter 2 in two dimensions, to decompose the solution X_γ into the sum $X_\gamma(x,t) = Z_\gamma(x,t) + V_\gamma(x,t)$ where Z_γ is the solution of the discrete linearized equation

$$Z_\gamma(x,t) \stackrel{\text{def}}{=} -\int_0^t \Delta_\epsilon\Delta_\gamma Z_\gamma(x,s)\,ds + M_\gamma(x,t) \tag{4.25}$$

and V_γ satisfies

$$\begin{aligned} V_\gamma(x,t) = X^0_\gamma(x) - \int_0^t \Delta_\epsilon\Delta_\gamma V_\gamma(x,s)\,ds \\ - A_\gamma\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X_\gamma(x,s)\,ds + \frac{B_\gamma}{3}\int_0^t \Delta_\epsilon K_\gamma *_\epsilon X^3_\gamma(x,s)\,ds \\ + \int_0^t \mathcal{U}_\gamma(x,s)ds + \int_0^t \mathsf{Err}_2(x,s)ds \end{aligned} \tag{4.26}$$

Remark 4.2.4 It is easy to see that we could have placed the contribution of the term $\mathcal{U}_\gamma(x,s)$ or the initial condition $X^0_\gamma(x)$ in the definition of the process Z_γ. The choice is essentially arbitrary, so we decided to deliver a complete treatment of the process Z_γ and postpone the discussion about the most problematic term $\mathcal{U}_\gamma(x,s)$ in the last sections.

Let now $P^{K,\gamma}_t = \exp\{-t\Delta_\gamma\Delta_\epsilon\}$ be the semigroup associated to the fourth order positive semidefinite operator $\Delta_\gamma\Delta_\epsilon$, as in the previous chapters, we will use notation $P^{K,\gamma}_t f$ to denote the application of the semigroup to the function $f : \Lambda_\varepsilon \to \mathbb{R}$. The semigroup is better described as a multiplicative operator in the Fourier coordinate

$$\widehat{P^{K,\gamma}_t f}(\omega) = \exp\left\{-t\alpha^{-1}[1-\cos(\pi\epsilon\omega)][1-\hat{K}_\gamma(\omega)]\right\}\hat{f}(\omega)$$

(see also Appendix B for further proprieties). Consider now the mild form of (4.26),

$$\begin{aligned} V_\gamma(x,t) = \int_0^t (-\Delta_\epsilon)P^{K,\gamma}_{t-s}K_\gamma *_\epsilon \left(A_\gamma X_\gamma(x,s) - \frac{B_\gamma}{3}X^3_\gamma(x,s)\right)\,ds \\ + P^{K,\gamma}_t X_\gamma(x,0) + \int_0^t P^{K,\gamma}_{t-s}\mathcal{U}_\gamma(x,s)ds + \int_0^t P^{K,\gamma}_{t-s}\mathsf{Err}_2(x,s)ds \end{aligned} \tag{4.27}$$

and it is easy to see that the last term is going to vanish deterministically in the limit as

$$\sup_{0\le t\le T}\left\|\int_0^t P^{K,\gamma}_{t-s}\mathsf{Err}_2(\cdot,s)ds\right\|_{L^\infty(\Lambda_\varepsilon)} \lesssim \int_0^T \left\|P^{K,\gamma}_{T-s}\nabla_\epsilon K_\gamma\right\|_{L^1(\Lambda_\varepsilon)}\gamma^{2/3}ds \le C(T)\gamma^{2/3}\,.$$

Before stating the main theorem of the chapter, we will present the assumption on the initial conditions. In the statement of the assumption we will extend a function defined on the discrete lattice Λ_ε to the continuous torus $\mathbb{T}$ via the extension operator defined in Section (1.10). Recall, for $a\in\mathbb{R}$, the definition of the Sobolev norms $H^a(\mathbb{T})$ and $\dot H^a(\mathbb{T})$

$$\|f\|^2_{H^a(\mathbb{T})} \stackrel{\text{def}}{=} \sum_{\omega\in\mathbb{Z}}\left(1+|\omega|^2\right)^a|\hat f(\omega)|^2 \qquad \|f\|^2_{\dot H^a(\mathbb{T})} \stackrel{\text{def}}{=} \sum_{\omega\in\mathbb{Z}}|\omega|^{2a}|\hat f(\omega)|^2\,.$$

In the following section, it will also be convenient to introduce a discrete version of the above norms (and seminorms) to measure the regularity in space of a function $f:\Lambda_\varepsilon\to\mathbb{R}$ defined on the discrete lattice

$$\|f\|^2_{H^a(\Lambda_\varepsilon)} \stackrel{\text{def}}{=} \sum_{\omega\in\Lambda_N}\left(1+|\omega|^2\right)^a|\hat f(\omega)|^2\,, \qquad \|f\|^2_{\dot H^a(\Lambda_\varepsilon)} \stackrel{\text{def}}{=} \sum_{\omega\in\Lambda_N}|\omega|^{2a}|\hat f(\omega)|^2\,.$$

Assumption 4.2.5 (Initial condition for Kawasaki dynamic) *For every $\gamma>0$ the Kawasaki dynamic is started from a deterministic configuration $\sigma^{(0)}$. Let*

$$X^0_\gamma(x) \stackrel{\text{def}}{=} \gamma^{-1/3}\sum_{z\in\Lambda_N}\kappa_\gamma(\epsilon^{-1}x-z)\sigma^{(0)}_z$$

The initial configuration satisfies the following proprieties:

- *There exists $\lambda>0$ such that the sequence of functions $X^0_\gamma:\Lambda_\varepsilon\to\mathbb{R}$ is bounded with respect to the*
$$\left\|X^0_\gamma\right\|_{H^{\frac12}(\Lambda_\varepsilon)} + \sum_{\omega\in\Lambda_N}|\omega|^\lambda|\hat X^0_\gamma(\omega)|$$
- *There exists $X^0\in\mathcal{C}(\mathbb{T})$ such that X^0 has zero average $\int_{\mathbb{T}}X^0(x)dx=0$ and*
$$\lim_{\gamma\to0}\left\|\mathsf{Ext}(X^0_\gamma)-X^0\right\|_{L^\infty(\mathbb{T})}=0\,.$$

We would like to remark that the choice of the norm is made out of convenience and not out of necessity. As it is easy to see using a Cauchy-Schwarz inequality, the above norm is controlled by $\left\|X_\gamma^0\right\|_{H^{\frac{1}{2}+\lambda}(\Lambda_\varepsilon)}$ from above, and (of course) by $\left\|X_\gamma^0\right\|_{H^{\frac{1}{2}}(\Lambda_\varepsilon)}$ from below. As we mentioned the main result of this chapter will be conditioned on the following conjecture.

Conjecture 4.2.6 *Let π be a probability over the spin configuration Σ_N and denote with $\mathbb{P}_\pi$ the law on the path space $\mathcal{D}([0,T],\Sigma_N)$ associated to the Kawasaki dynamic started from the initial measure π. Then, for all $b>0$, and for any π*

$$\lim_{\gamma\to 0}\mathbb{P}_\pi\left(\sup_{0\le t\le T}\sup_{x\in\Lambda_\varepsilon}\left|\int_0^t P_{t-s}^{K,\gamma}\mathcal{U}_\gamma(x,s)ds\right|>b\right)=0 \tag{4.28}$$

We are now ready to state the main result.

Theorem 4.2.7 (Conditioned on Conjecture 4.2.6) *Assume the statement of Conjecture 4.2.6. Under Assumption 4.2.5, the process X_γ converges in law in $\mathcal{D}([0,T],\mathcal{C}(\mathbb{T}))$ to the solution of the stochastic Cahn-Hilliard equation*

$$dX(x,t)=-\Delta\left(\Delta X-X^3+AX\right)(x,t)\,ds+\sqrt{2}\zeta(x,t) \tag{4.29}$$

started from X^0, where the noise is a stationary Gaussian process with covariance

$$\mathbb{E}\left[\zeta(\varphi),\zeta(\psi)\right]=\int_{\mathbb{T}\times[0,T]}\nabla\varphi(x,t)\nabla\psi(x,t)\,dx\,dt$$

The plan of the rest of the chapter is the following:

- In Section 4.3 we will recall the existence and uniqueness theory of the solutions of (4.8).
- In Section 4.4 we are going to show the convergence in distribution of Z_γ defined in (4.25) to the solution of the linear part of (4.8)

  $$dZ(x,t)=-\Delta\Delta Z(x,t)+\sqrt{2}\zeta(x,t)$$

  in the topology of $\mathcal{D}([0,T];\mathcal{C}(\mathbb{T}))$ where the law of ζ is described after (4.8). This result is conditionally on the Conjecture 4.2.6.
- In Section 4.5 we are going to show, under the Conjecture 4.2.6 the convergence of the discrete remainder V_γ to the solution of the nonlinear random PDE

  $$\partial_t V(x,t)=-\Delta\Big(\Delta V(x,t)+F(V(x,t)+Z(x,t))\Big)$$

  for the polynomial $F(x)=Ax-\frac{1}{3}x^3$.

- In the last section we are going to bring some evidence towards Conjecture 4.2.6, based on a second order Boltzmann-Gibbs principle, first introduced in [GJ14], and on a modification of the usual 1-block/2-blocks argument.

4.3 Stochastic Cahn-Hilliard equation

We will now briefly describe some of the aspects of the stochastic Cahn-Hilliard equation in dimension one

$$\begin{cases} dX & = -\Delta\left(\Delta X + F(X)\right) + \sqrt{2}\zeta \\ X(0) & = X^0 \in \mathcal{C}(\mathbb{T}) \end{cases} \tag{4.30}$$

where $\mathcal{C}(\mathbb{T})$ is the set of continuous function on the torus. The Gaussian noise ζ is white in time and coloured in space with covariance structure given by

$$\mathbb{E}\left[\zeta(\phi)\zeta(\psi)\right] = \int_{[0,T]\times\mathbb{T}} \nabla\phi(t,x)\nabla\psi(t,x)\, dt$$

The function $F : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree 3 with negative leading coefficient. By solution to (4.30) we mean a process adapted to the filtration generated by the noise and satisfying, for $(x,t) \in \mathbb{T} \times [0,T]$

$$X(t,x) = \int_0^t \Delta P_{t-s}^K F(X(s,\cdot))(x)ds + \sqrt{2}\int_0^t P_{t-s}^K \zeta(ds,x) \tag{4.31}$$

where P_t^K is the convolution semigroup defined as

$$P_t^K(x) = \frac{1}{2}\sum_{\omega\in\mathbb{Z}} e^{-\pi^4 t|\omega|^4} e_\omega(x)$$

The last integral of (4.31) is the stochastic convolution, defined for example in [DPZ14, Sec. 5].

The stochastic Cahn-Hilliard equation has been studied in [DPD96, CW01] for a noise which is white both in time and in space in $d = 1$. The same techniques, can be used to prove existence and uniqueness of the solution of (4.31)

Theorem 4.3.1 *For all $T > 0$, there exists a unique solution to* (4.30) *in* $\mathcal{C}([0,T];\mathcal{C}(\mathbb{T}))$.

In this section we are going to prove Theorem 4.3.1 and we are also going to show some continuity of the problem with respect to the polynomial F.

In order to state the results we need to introduce some notations that will be used later. Following [DPD96], we let Z to be the solution in $[0,T]\times\mathbb{T}$ of the linearization of (4.30)

$$\begin{cases} dZ &= -\Delta^2 Z + \sqrt{2}\zeta \\ Z(0) &= 0 \end{cases} \tag{4.32}$$

From [DPZ14, Theorem 5.2] and the fact that

$$\int_0^t \left\|\nabla P^K_{t-s}\right\|^2_{L^2(\mathbb{T})} ds < \infty$$

we have that (4.32) has a unique continuous Gaussian solution given by the stochastic convolution

$$\sqrt{2}\int_0^t P^K_{t-s}\zeta(ds,x)\,.$$

It is convenient to define a distance on $[0,T]\times\mathbb{T}$ compatible with the regularizing effect of the semigroup P^K. For $(t,x)\neq(s,y)\in[0,T]\times\mathbb{T}$, define the distance

$$\mathrm{dist}_{[0,T]\times\mathbb{T}}\left((t,x),(s,y)\right) = |t-s|^{\frac{1}{4}} + |x-y|$$

and denote with $\mathcal{H}^\alpha([0,T]\times\mathbb{T})$ the Hölder space defined with the above distance.
The next proposition is proven in [DPD96, Proposition 1.2].

Proposition 4.3.2 *For all $\kappa>0$, the process Z lies in the Hölder space $\mathcal{H}^{\frac{1}{2}-\kappa}([0,T]\times\mathbb{T})$*

The solution to the original problem can be decomposed as $X(t)=Z(t)+V(t)$ where V is the mild solution of

$$\begin{cases} \partial_t V(t,x) &= -\Delta^2 V(t,x) - \Delta F\left(Z(t,x)+V(t,x)\right) \\ V(0) &= X^{(0)}\,. \end{cases} \tag{4.33}$$

We will show later that the differential equations (4.30) and (4.32) preserve the space average of the solution, because the noise is not acting on the zeroth Fourier mode. This means that, at the cost of modifying $F(\cdot)$ with its translation by the average of $X^{(0)}$, we can assume the initial data to satisfy $\int_{\mathbb{T}} X^{(0)}(x)dx = 0$. In the space $\dot{\mathcal{S}}(\mathbb{T})$ of smooth functions with zero average the collection $\|\cdot\|_{\dot{H}^a(\mathbb{T})}$ is a family of norms and the operator Δ is invertible.

In the following proposition $\tilde{Z}$ denotes a generic process.

Proposition 4.3.3 *Let processes $\tilde{Z}$ be in $L^\infty([0,T]\times\mathbb{T})$ and let $V_0\in L^\infty(\mathbb{T})$. Then for all*

$\kappa > 0$ *there exists a Lipschitz continuous map*

$$\mathcal{S}_T^K : L^\infty([0,T] \times \mathbb{T}) \times L^\infty(\mathbb{T}) \longrightarrow \mathcal{C}([0,T]; H^{2-\kappa}(\mathbb{T}))$$

that associates to the process $\tilde{Z}$ *and initial condition* V_0*, the solution of* (4.33)*, started from* V_0.

The proof of the above proposition is taken essentially from [DPD96, CW01]. In those articles the authors considered the stochastic Cahn-Hilliard equation driven by a space time white noise, using the same Da Prato-Debussche decomposition. While the proof in [DPD96] relies on the differentiability of the Gaussian process that solves the linearized equation, the method in [CW01] only uses the fact that it is bounded in $L^\infty([0,T] \times \mathbb{T})$ and therefore it applies also to our case. Before the proof we quote a proposition [DPD96, Proposition 2.1], restated in case F is a polynomial of degree 3 with negative leading coefficient.

Proposition 4.3.4 *Let* F *be defined above and consider* $v_0, u_0 \in H^{-1}(\mathbb{T})$ *and functions* $g, h \in L^4([0,T] \times \mathbb{T})$*. Assume that* $v, u \in L^2([0,T]; H^1) \cap L^4([0,T] \times \mathbb{T})$ *solve the following PDE*

$$\begin{cases} \partial_t u(t) & = -\Delta^2 u(t) - \Delta F(u(t) + h(t)) \\ u(0) & = u_0 \end{cases}$$
$$\begin{cases} \partial_t v(t) & = -\Delta^2 v(t) - \Delta F(v(t) + g(t)) \\ v(0) & = v_0 \end{cases}$$

Then there exists a constant $C = C(T, g, h)$ *such that, for* $t \in [0,T]$

$$\begin{aligned} \|v(t) - u(t)\|_{H^{-1}}^2 &\leq C \|v_0 - u_0\|_{H^{-1}}^2 \\ &\quad + C \int_{[0,T]\times\mathbb{T}} |g(t,x) - h(t,x)| \left(|u+h|^3(t,x) + |v+g|^3(t,x) \right) dx \\ &\leq C \Big[\|v_0 - u_0\|_{H^{-1}}^2 + \|g - h\|_{L^4([0,T]\times\mathbb{T})} \\ &\quad \times \Big(\|g\|_{L^4([0,T]\times\mathbb{T})}^3 + \|h\|_{L^4([0,T]\times\mathbb{T})}^3 + \|v\|_{L^4([0,T]\times\mathbb{T})}^3 + \|u\|_{L^4([0,T]\times\mathbb{T})}^3 \Big) \Big] \end{aligned}$$

We will also need a lemma concerning the smoothing proprieties of the semigroup P^K in case of dimension one, the lemma is [CW01, Lemma 1.6].

Lemma 4.3.5 *For any* $1 \leq p \leq q \leq \infty$ *then there exists a constant* $C = C(p,q)$ *such that* *for any function* $f \in L^\infty([0,T];L^p(\mathbb{T}^d))$

$$\left\| \int_s^t P^K_{t-r} f(r)\, dr \right\|_{L^q(\mathbb{T}^d)} \le C \int_s^t (t-s)^{-\frac{d}{4}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f(r)\|_{L^p(\mathbb{T}^d)}\, dr \tag{4.34}$$

$$\left\| \int_s^t \Delta P^K_{t-r} f(r) dr \right\|_{L^q(\mathbb{T}^d)} \le C \int_s^t (t-s)^{-\frac{d}{4}\left(2+\frac{1}{p}-\frac{1}{q}\right)} \|f(r)\|_{L^p(\mathbb{T}^d)}\, dr \tag{4.35}$$

For $b \le a$ *then there exists a constant* $C = C(a,b)$ *such that for any function* $f \in L^\infty([0,T];H^b(\mathbb{T}^d))$

$$\left\| \int_s^t P^K_{t-r} f(r)\, dr \right\|_{H^a(\mathbb{T}^d)} \le C \int_s^t (t-s)^{-\frac{1}{4}(a-b)} \|f(r)\|_{H^b(\mathbb{T}^d)}\, dr \tag{4.36}$$

$$\left\| \int_s^t \Delta P^K_{t-r} f(r) dr \right\|_{H^a(\mathbb{T}^d)} \le C \int_s^t (t-s)^{-\frac{1}{4}(2+a-b)} \|f(r)\|_{H^b(\mathbb{T}^d)}\, dr \tag{4.37}$$

Proof. We will first show (4.34). Using Minkowski's inequality and the Young inequality for the convolution, with $1/q + 1 = 1/p + 1/k$

$$\left\| \int_s^t P^K_{t-r} f(r)\, dr \right\|_{L^q(\mathbb{T}^d)} \le \int_s^t \left\| P^K_{t-r} \right\|_{L^k(\mathbb{T}^d)} \|f(r)\|_{L^p(\mathbb{T}^d)}\, dr$$

and therefore it is sufficient to show that $\left\| P^K_{t-r}(\cdot) \right\|_{L^k(\mathbb{T}^d)} \le C(k)(t-r)^{-\frac{d}{4}\left(1-\frac{1}{k}\right)}$. From the fact that $\|g\|^k_{L^k} \le \|g\|^{k-1}_{L^\infty} \|g\|_{L^1}$ it is possible to prove the result for $k=1$ and $k=\infty$. Assume $k=1$. The result is based on the calculation of the $L^1(\mathbb{R}^d)$ norm of G_t, the Green function for the problem in the whole space $\mathbb{R}^d$

$$\partial_t G_t(x) = -\Delta\Delta G_t(x) + \delta_0 \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}$$

and the fact that $P^K_t(x) = \sum_{i\in\mathbb{Z}^d} G_t(x+i)$ that implies $\left\| P^K_t \right\|_{L^k(\mathbb{T}^d)} = \|G_t\|_{L^k(\mathbb{R}^d)}$ for $1 \le k < \infty$. From the reverse Fourier transform we have

$$G_t(x) = \int_{\mathbb{R}^d} e^{-\pi^4 t|\xi|^4 - \pi i x\cdot\xi} d\xi = t^{-\frac{d}{4}} \int_{\mathbb{R}^d} e^{-\pi^4|\xi|^4 - \pi i t^{-\frac{1}{4}} x\cdot\xi} d\xi$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} |G_t(x)| dx = \int_{\mathbb{R}^d} \left| t^{-\frac{d}{4}} \int_{\mathbb{R}^d} e^{-\pi^4|\xi|^4 - \pi i t^{-\frac{1}{4}} x\cdot\xi} d\xi \right| dx \\ = \int_{\mathbb{R}^d} t^{\frac{d}{4}} \left| t^{-\frac{d}{4}} \int_{\mathbb{R}^d} e^{-\pi^4|\xi|^4 - \pi i x\cdot\xi} d\xi \right| dx\,.\end{aligned}$$

We show now that
$$\left\|\int_{\mathbb{R}^d} e^{-\pi^4|\xi|^4-\pi i x\cdot\xi} d\xi\right\|_{L^1(\mathbb{R}^d)} < \infty$$
using the same idea as [BCD11, Lemma 2.4]. Use the summation by parts to write

$$\begin{aligned}\int_{\mathbb{R}^d} |G_t(x)| dx &= \int_{\mathbb{R}^d} (1+|x|^2)^{-d} \left|\int_{\mathbb{R}^d} (1+|x|^2)^d e^{-\pi^4|\xi|^4} e^{-\pi i x\cdot\xi} d\xi\right| dx \\ &= \int_{\mathbb{R}^d} (1+|x|^2)^{-d} \left|\int_{\mathbb{R}^d} e^{-\pi^4|\xi|^4} (\mathrm{Id} + \Delta_\xi)^d e^{-\pi i x\cdot\xi} d\xi\right| dx \\ &= \int_{\mathbb{R}^d} (1+|x|^2)^{-d} \left|\int_{\mathbb{R}^d} e^{-\pi i x\cdot\xi} (\mathrm{Id} + \Delta_\xi)^d e^{-\pi^4|\xi|^4} d\xi\right| dx\,.\end{aligned}$$

Since $\left|(\mathrm{Id} + \Delta_\xi)^d e^{-\pi^4|\xi|^4}\right| \lesssim (1+|\xi|^{2d}) e^{-\pi^4|\xi|^4} \lesssim e^{-c|\xi|^4}$, it follows that $\|G_t\|_{L^1(\mathbb{R}^d)} < \infty$. Assume now that $k=\infty$. In this case, by a change of variable

$$\|P_t^K\|_{L^\infty(\mathbb{T}^d)} \lesssim \sum_{\omega\in\mathbb{Z}^d} e^{-\pi^4 t|\omega|^4} = \left\|P_{t/2}^K\right\|_{L^2(\mathbb{T}^d)}^2$$

and the same proof as in the case $k=1$ shows that $\left\|P_{t/2}^K\right\|_{L^2(\mathbb{T}^d)} \lesssim t^{-\frac{d}{8}}$. Putting together the above estimates we obtain

$$\|P_t^K\|_{L^k(\mathbb{T}^d)} \lesssim t^{-\frac{d}{4}\left(1-\frac{1}{k}\right)}\,.$$

In a similar way one proves (4.35). We are now going to prove (4.36), and (4.36) will follows from a different choice of a, b. We will use the Fourier series expansion of the semigroup and the fact that $\sup_{x\geq 0} x^{\frac{a-b}{4}} e^{-cx^4} \leq C(a,b)$

$$\begin{aligned}\left\|\int_s^t P_{t-r}^K f(r)\, dr\right\|_{H^a(\mathbb{T}^d)} &\lesssim \int_s^t \left(\sum_{\omega\in\mathbb{Z}^d} |\omega|^{2a} e^{-2(t-r)\pi^4|\omega|^4} |\hat{f}(r,\omega)|^2\right)^{\frac{1}{2}} dr \\ &\lesssim \int_s^t (t-s)^{-\frac{a-b}{4}} \|f\|_{H^b(\mathbb{T}^d)}\, dr\,.\end{aligned}$$

□

We will now prove Proposition 4.3.3.

Proof of Proposition 4.3.3. The proof is divided into three parts: in the first part we prove existence and uniqueness of the solution as a fixed point problem for small times, in the second part we prove some a priori estimates for the process in a weaker norm and in the last part we will prove the global existence and the continuity of the map.

Since there is no possibility of confusion we will use L^p for $L^p(\mathbb{T})$ and similarly for the Sobolev spaces H^a.
Consider the decomposition $X(t) = \tilde{Z}(t) + \tilde{V}(t)$ where $\tilde{Z}(t)$ is the solution of (4.32) with initial condition X^0 and $\tilde{V}$ the solution of

$$\begin{cases} \partial_t \tilde{V}(t,x) & = -\Delta^2 \tilde{V}(t,x) - \Delta F\left(\tilde{Z}(t,x) + \tilde{V}(t,x)\right) \\ \tilde{V}(0) & = 0\,. \end{cases} \tag{4.38}$$

The choice to put the contribution of the initial condition in $\tilde{Z}$ has been made just for convenience.
By assumption the initial condition satisfies $\left\|X^0\right\|_{L^\infty} < \infty$ hence

$$\|\tilde{Z}\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \le \sup_{t>0} \|P_t^K\|_{L^\infty \to L^\infty} \left\|X^0\right\|_{L^\infty} + \|Z\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \ .$$

For reasons that will be clear later, we would like to have a bound in terms of $\left\|X^0\right\|_{L^2}$ and therefore, using $\left\|P_t^K\right\|_{L^2 \to L^\infty} \lesssim t^{-\frac{1}{8}}$ as proven in the proof of Lemma 4.3.5,

$$\|\tilde{Z}\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \lesssim t^{-\frac{1}{8}} \left\|X^0\right\|_{L^2} + \|Z\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \ .$$

where Z is the solution of (4.32). Conditioning on Z, we consider the mild form of (4.38)

$$\tilde{V}(t,x) = \int_0^t P_{t-s}^K (-\Delta) F\left(\tilde{Z} + \tilde{V}\right)(s,x)\, ds\,.$$

Using (4.35) we have that there exists a constant C, depending only on $\left\|Z\right\|_{L^\infty([0,T]\times\mathbb{T})}$ and $\left\|X^0\right\|_{L^2(\mathbb{T})}$,

$$\begin{aligned} \left\|\tilde{V}(t)\right\|_{L^\infty} \le \left\|\int_0^t P_{t-s}^K \Delta F\left(\tilde{Z} + \tilde{V}\right)(s)\, ds\right\|_{L^\infty} \\ \le C \int_0^t (t-s)^{-\frac{1}{2}} \left\|F\left(\tilde{Z} + \tilde{V}\right)\right\|_{L^\infty}(s)\, ds \\ \le C \int_0^t (t-s)^{-\frac{1}{2}} \left(s^{-\frac{1}{8}} + 1 + \left\|\tilde{V}(s)\right\|_{L^\infty}\right)^3 ds \end{aligned}$$

Therefore, there exists a time $T^* = T^*\big(\left\|Z\right\|_{L^\infty([0,T]\times\mathbb{T})}, \left\|X^0\right\|_{L^2}\big)$ in such a way that the map

$$v \mapsto \int_0^t P_{t-s}^K \Delta F\left(\tilde{Z} + v\right)(s)\, ds$$

is a contraction in the ball of radius 1 in the Banach space $\mathcal{C}([0,T^*]; L^\infty(\mathbb{T}))$.
It is easy to see, using Morrey's inequality and (4.37), that the solution is actually continuous

in space.
We will now provide a global control over the L^2-norm of the remainder process $\tilde{V}(t)$. In order to do this consider the Galerkin approximation V^M the solution to

$$\begin{cases} \partial_t V^M(t,x) & = -\Delta^2 V^M(t,x) - \Pi^M \Delta F\left(\tilde{Z}(t,x) + V^M(t,x)\right) \\ V^M(0) & = \Pi^M V_0 \,. \end{cases} \tag{4.39}$$

From the assumption of the theorem

$$\|\tilde{Z}\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} < \infty, \qquad V_0 \in L^2(\mathbb{T}) \,.$$

and then V^M is a well defined smooth function for every finite M. We will first find some a priori bounds for V^M with respect to the norm $L^\infty([0,T];H^{-1})$.
Differentiating with respect to t the quantity $\int_{\mathbb{T}} V^M(t,x)dx = 0$, it is immediate to see that the average of $V(t)$ is constant in time, and in particular is equal to 0. In this case $\|V^M\|_{H^{-1}}^2 = \langle V^M, (-\Delta)^{-1} V^M \rangle$ and, using integration by parts on the periodic torus, and the fact that Π^M is autoadjoint we have

$$\frac{1}{2}\partial_t \|V^M(t)\|_{H^{-1}}^2 = -\|V^M(t)\|_{H^1}^2 + \left\langle F\left(\tilde{Z}(t) + V^M(t)\right), V^M(t)\right\rangle \,.$$

The last quantity is equal to

$$\left\langle F\left(\tilde{Z}(t) + V^M(t)\right), \tilde{Z}(t) + V^M(t)\right\rangle - \left\langle F\left(\tilde{Z}(t) + V^M(t)\right), \tilde{Z}(t)\right\rangle$$

and we use the assumptions over F to bound it with

$$-c_1 \left\| \left(\tilde{Z}(t) + V^M(t)\right)^4 \right\|_{L^1} + C_1 + C_2 + c_2 \left\| \left|\tilde{Z}(t) + V^M(t)\right|^3 \tilde{Z}(t) \right\|_{L^1}$$

an application of the inequality $a^3 b \leq \frac{1}{4}a^4 + b^4$ to the last term yields

$$\partial_t \|V^M(t)\|_{H^{-1}}^2 + \|V^M(t)\|_{H^1}^2 + \frac{c_1}{2}\left\|\tilde{Z}(t) + V^M(t)\right\|_{L^4}^4 \leq c\|\tilde{Z}(t)\|_{L^4}^4 + C \,.$$

Integrating the inequality above, one can see that V^M is uniformly bounded in $L^\infty([0,T];H^{-1})$, $L^2([0,T];H^1)$ and $L^4([0,T]\times\mathbb{T})$

$$\sup_{t\leq T}\|V^M(t)\|_{H^{-1}}^2 + \int_0^T \|V^M(s)\|_{H^1}^2\,ds + \int_0^T \|V^M(s)\|_{L^4}^4\,ds \lesssim \|V_0\|_{H^{-1}}^2 + \|\tilde{Z}\|_{L^4([0,T];L^4(\mathbb{T}))}^4 + 1 \,. \tag{4.40}$$

We are going to need stronger a priori bounds, therefore we differentiate in t the L^2 norm of V^M. Using (4.39) and integrating by parts we obtain

$$\frac{1}{2}\partial_t \left\|V^M\right\|_{L^2}^2 = -\left\|\Delta V^M\right\|_{L^2}^2 - \left\langle F\left(\tilde{Z}(t) + V^M(t)\right), \Delta V^M(t)\right\rangle$$

We will now rewrite the last term as the sum

$$-\left\langle F\left(V^M(t)\right), \Delta V^M\right\rangle - \left\langle F\left(\tilde{Z}(t) + V^M(t)\right) - F\left(V^M(t)\right), \Delta V^M(t)\right\rangle .$$

We can bound the first term, integrating by parts and using assumption $F'(x) \leq -c_3 x^2 + c_4$

$$\left\langle F'\left(V^M(t)\right)\nabla V^M(t), \nabla V^M(t)\right\rangle \leq -c_3\left\|V^M(t)\nabla V^M(t)\right\|_{L^2}^2 + c_4\left\|\nabla V^M(t)\right\|_{L^2}^2 .$$

The second term can be bounded using $|F(z+v) - F(v)| \leq C_3|z|(1+z^2+v^2)$,

$$\begin{aligned}
&\left\langle F\big(\tilde{Z}(t) + V^M(t)\big) - F\left(V^M(t)\right), \Delta V^M(t)\right\rangle \\
\leq &C_3\int_{\mathbb{T}}\left(1 + |V^M(t,x)|^2 + |\tilde{Z}(t,x)|^2\right)|\tilde{Z}(t,x)\Delta V^M(t,x)|dx \\
\leq &C_3\int_{\mathbb{T}}\left(1 + |\tilde{Z}(t,x)|^2\right)|\tilde{Z}(t,x)\Delta V^M(t,x)|dx \\
+&C_3\big\|\tilde{Z}(t)\big\|_{L^\infty}\int_{\mathbb{T}}|V^M(t,x)|^2|\Delta V^M(t,x)|dx \\
\leq &\frac{1}{2}\left\|\Delta V^M(t)\right\|_{L^2}^2 + \frac{C_3^2}{2}\left(\big\|\tilde{Z}(t)\big\|_{L^2}^2 + \big\|\tilde{Z}(t)\big\|_{L^6}^6 + \big\|\tilde{Z}(t)\big\|_{L^\infty}^2\left\|V^M(t)\right\|_{L^4}^4\right)
\end{aligned}$$

where in the last line we used the Young inequality $ab \leq \frac{1}{4C_3}a^2 + C_3 b^2$. Collecting the bounds together we have that

$$\begin{aligned}
\frac{1}{2}\sup_{t\leq T}\left\|V^M(t)\right\|_{L^2}^2 + \frac{1}{2}\int_0^T\left\|V^M(s)\right\|_{H^2}^2 ds + c_3\int_0^T\left\|V^M(s)\nabla V^M(s)\right\|_{L^2}^2 ds \\
\leq c_4\int_0^T\left\|\nabla V^M(s)\right\|_{L^2}^2 ds + \frac{C_3^2}{2}\big\|\tilde{Z}\big\|_{L^\infty([0,T]\times\mathbb{T})}^2\int_0^T\left\|V^M(s)\right\|_{L^4}^4 ds \\
+\frac{1}{2}\|V_0\|_{L^2}^2 + C\int_0^T\big\|\tilde{Z}(s)\big\|_{L^6}^6 ds + 1 .
\end{aligned}$$

From the a priori estimates obtained in (4.40), we can bound the right-hand-side of the above equation with a constant times

$$1+\|V_0\|_{L^2}^2+\int_0^T \left\|\tilde{Z}(s)\right\|_{L^6}^6 ds$$
$$+\left(1+\left\|\tilde{Z}\right\|_{L^\infty([0,T]\times\mathbb{T})}^2\right)\left(1+\|V_0\|_{H^{-1}}^2+\int_0^T \left\|\tilde{Z}(s)\right\|_{L^4}^4 ds\right) .$$

We now upgrade the previous estimates with an interpolation and the Sobolev's inequality: for all $r\geq 1$

$$\int_0^T \left\|V^M(s)\right\|_{L^r}^2 ds \lesssim \int_0^T \left\|V^M(s)\right\|_{H^1}^2 ds .$$

Let $2\leq q$ and $a\geq 2$. We now would like to interpolate L^q in L^2 and L^r for r big enough. Let $q=2\lambda+(1-\lambda)r$

$$\begin{aligned}\int_0^T \left\|V^M(s)\right\|_{L^q}^a ds &\leq \int_0^T \left\|V^M(s)\right\|_{L^2}^{a\lambda}\left\|V^M(s)\right\|_{L^r}^{a(1-\lambda)} ds\\ &\leq \sup_{0\leq t\leq T}\left\|V^N(t)\right\|_{L^2}^{a-2}\int_0^T \left\|V^M(s)\right\|_{L^r}^2 ds \quad (4.41)\end{aligned}$$

where we chose $\lambda=1-\frac{2}{a}$ and r consequently. Therefore V^M is bounded in $L^a([0,T];L^q(\mathbb{T}))$ uniformly in M for any $0<q,a<\infty$. We therefore obtain that the sequence V^M is bounded in $L^2([0,T];H^2(\mathbb{T}))$ and, by the Banach-Alaoglu theorem, it has a subsequence converging in the weak-$*$ topology to a limit which will be called V in $L^2([0,T];H^2(\mathbb{T}))$. Moreover by a Fourier decomposition, it is possible to see that, for any $\kappa>0$, the sequence is converging strongly to V in $L^2([0,T];H^{2-\kappa}(\mathbb{T}))$ and, by the Sobolev inequality [Bre11, Theorem 8.8] in $L^2([0,T];\mathcal{C}(\mathbb{T}))$.
Therefore, using (4.41) we have that the limit V is a weak solution of (4.38). By the first part of the proof the solution to (4.38) exists and it is unique locally. Moreover

$$\sup_{t\leq T}\limsup_{M\to\infty}\left\|V^M(t)\right\|_{L^2}^2<\infty\,, \qquad \int_0^T \limsup_{M\to\infty},\left\|V^M(t)\right\|_{L^q}^a<\infty \qquad (4.42)$$

where the constants only depend polynomially on $\left\|\tilde{Z}\right\|_{L^\infty([0,T]\times\mathbb{T})}$ and $\|V_0\|_{L^2}$. This implies the existence and uniqueness of the solution in $[0,T]$, because of the form of the stopping time.

We are going to show now that the map $\mathcal{S}_T$ is locally Lipschitz continuous. Let Z_1,Z_2 two elements of $L^\infty([0,T];L^\infty(\mathbb{T}))$ and $X_1^0,X_2^0\in L^\infty(\mathbb{T})$ such that

$$\left\|Z_i\right\|_{L^\infty([0,T];L^\infty(\mathbb{T}))}\leq R\,, \qquad \left\|X_i^0\right\|_{L^\infty(\mathbb{T})}<R \qquad \text{for } i=1,2 .$$

Let

$$\tilde{Z}_i = P_t^K X_i^0 + Z_i \qquad \text{for } i = 1, 2$$

and $\tilde{V}_1, \tilde{V}_2$ be the solutions of (4.38) driven by the processes $\tilde{Z}_1$ and $\tilde{Z}_2$ respectively. An application of (4.35) shows that

$$\begin{aligned}
&\|\tilde{V}_1(t) - \tilde{V}_2(t)\|_{L^\infty} \\
&\quad \leq \left\| \int_0^t P_{t-s}^K \Delta \left\{ F\big(\tilde{Z}_1 + \tilde{V}_1\big)(s) - F\big(\tilde{Z}_2 + \tilde{V}_2\big)(s) \right\} \, ds \right\|_{L^\infty} \\
&\quad \leq \int_0^t (t-s)^{-\frac{3}{4}} \left\| F\big(\tilde{Z}_1 + \tilde{V}_1\big)(s) - F\big(\tilde{Z}_2 + \tilde{V}_2\big)(s) \right\|_{L^1} \, ds
\end{aligned}$$

and using the fact that F has degree 3 and (4.42) we can bound the previous quantity with

$$\begin{aligned}
&C(R) \int_0^t (t-s)^{-\frac{3}{4}} \left(\big\|\tilde{Z}_1(s) - \tilde{Z}_2(s)\big\|_{L^\infty} + \left\|\tilde{V}_1(s) - \tilde{V}_2(s)\right\|_{L^\infty} \right) \, ds \\
&\leq C(R) T^{\frac{1}{4}} \big\|\tilde{Z}_1 - \tilde{Z}_2\big\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} + C(R) \int_0^t (t-s)^{-\frac{3}{4}} \left\|\tilde{V}_1(s) - \tilde{V}_2(s)\right\|_{L^\infty} \, ds
\end{aligned}$$

By a version of the Grönwall inequality in Lemma 5.7 of [HW13] we have that, there exists C, depending on R and T such that, for $0 \leq t \leq T$

$$\begin{aligned}
\left\|\tilde{V}_1(t) - \tilde{V}_2(t)\right\|_{L^\infty} &\leq C(R,T) \big\|\tilde{Z}_1 - \tilde{Z}_2\big\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \\
&\leq C(R,T) \left(\|X_1 - X_2\|_{L^\infty(\mathbb{T})} + \big\|Z_1 - Z_2\big\|_{L^\infty([0,T];L^\infty(\mathbb{T}))} \right)
\end{aligned}$$

□

4.4 The linearized equation

In this section we will prove the tightness for the laws of the processes Z_γ satisfying for $(x,t) \in \Lambda_\varepsilon \times [0,T]$ the linearized version of (4.24) which is given by

$$Z_\gamma(x,t) = -\int_0^t \Delta_\epsilon \Delta_\gamma Z_\gamma(x,s) \, ds + M_\gamma(x,t) \, . \tag{4.43}$$

Moreover we are going to show that each limiting law is the law of the solution of

$$\begin{cases} dZ & = -\Delta^2 Z + \sqrt{2}\zeta \\ Z(0) & = 0 \, . \end{cases} \tag{4.44}$$

The tightness for the sequence of laws has to be proven in a common space. We will therefore extend Z_γ, defined on the discrete torus Λ_ε, to the whole torus $\mathbb{T}$ using the extension operator Ext defined in the previous chapters. For this section we will still denote with Z_γ the extended process.

Theorem 4.4.1 (Tightness of Z_γ) *The sequence of laws of Z_γ, defined as the solution of* (4.43)*, is tight in* $\mathcal{D}([0,T];\mathcal{C}(\mathbb{T}))$*. More precisely, for all $p>0$ we have*

$$\limsup_{\gamma\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\|Z_\gamma\|^p_{L^\infty(\mathbb{T})}\right]<\infty$$

Proof. The proof will follow the lines of [MW17a, Sec. 5]. Let us introduce for $0\le s<t$ and $x\in\Lambda_\varepsilon$, the approximating martingale process

$$R_{\gamma,t}(s,x)\stackrel{\text{def}}{=}\int_0^s P^{K,\gamma}_{t-r}dM_\gamma(r,\cdot)(x)\ . \tag{4.45}$$

The above quantity can be also written as

$$\begin{aligned}R_{\gamma,t}(s,x) &=\\ \int_{[0,s)}\epsilon\sum_{z\in\Lambda_N}P^{K,\gamma}_{t-r}(x-\epsilon z)&\sum_{u\in\Lambda_N}\left(\kappa_\gamma(u-z)-\kappa_\gamma(u+1-z)\right)(\sigma_{u+1}-\sigma_u)(\alpha^{-1}r^-)\\ &\times\delta^{-1}\left(\mathcal{J}^{u,u+1}_{\alpha^{-1}r}(\sigma)-\alpha^{-1}c^K_\gamma(u,u+1,\sigma(\alpha^{-1}r^-))dr\right)\end{aligned}\tag{4.46}$$

where $\mathcal{J}^{u,v}_{\alpha^{-1}r}(\sigma)$ is a Poisson point process with intensity $\alpha^{-1}c^K_\gamma(u,v,\sigma(\alpha^{-1}r^-))$, governing the times at which the spins in positions u,v are exchanged. We will also denote with $R_{\gamma,t}(s,\varphi)$ the (discrete) convolution $\sum_z\epsilon R_{\gamma,t}(s,z)\varphi(x-z)$ with a function $\varphi:\Lambda_\varepsilon\to\mathbb{R}$. From the above description it is possible to see that for fixed t, $R_{\gamma,t}(s,\varphi)$ is a martingale in $0\le s\le t$ with quadratic variation (4.45) given by

$$\begin{aligned}\langle R_{\gamma,t}(\cdot,\varphi)\rangle_s &=\int_0^s\sum_{y_1,y_2\in\Lambda_\varepsilon}\epsilon^2P^{K,\gamma}_{t-r}\varphi(y_1)P^{K,\gamma}_{t-r}\varphi(y_2)d\left\langle M_\gamma(\cdot,y_1),M_\gamma(\cdot,y_2)\right\rangle_r\\ &\le 8\int_0^s\sum_{z\in\Lambda_\varepsilon}\epsilon\left(P^{K,\gamma}_{t-r}\nabla_\epsilon K_\gamma*_\epsilon\varphi(z)\right)^2dr\lesssim\sum_{\omega\in\Lambda_N\setminus\{0\}}\frac{|\hat K_\gamma(\omega)|^2|\hat\varphi(\omega)|^2}{\epsilon^{-2}\gamma^2(1-\hat K_\gamma(\omega))}\end{aligned}\tag{4.47}$$

where we used (4.19) and the fact that $|c^K_\gamma(u,v,\sigma)|\le 2$ uniformly in its parameters. It is easy to see, using $\varphi\equiv 1$, that the space average of the process $R_{\gamma,t}$ is constantly equal to 0. In Lemma 4.4.2 it is proven that, for any $0<a<1/2$ and $p\ge 2$, for $x\in\mathbb{T}$ and $0\le s\le t$

$$\mathbb{E}\left[|Z_\gamma(t,x)-R_{\gamma,t}(s,x)|^p\right]^{\frac{1}{p}}\lesssim|t-s|^{\frac{a}{4}}+\gamma^{\frac{4}{3}}\ .$$

It follows that there exists a constant $C = C(a, p)$ such that for any $x, y \in \mathbb{T}$ and $0 \le s \le t \le T$ we have

$$\sup_{t \le T,\, x \in \mathbb{T}} \mathbb{E}\left[|Z_\gamma(t,x)|^p\right]^{\frac{1}{p}} \le C \tag{4.48}$$
$$\mathbb{E}\left[|Z_\gamma(t,x) - Z_\gamma(s,y)|^p\right]^{\frac{1}{p}} \le C|t-s|^{\frac{a}{4}} + C|x-y|^a + C\gamma^{\frac{4}{3}}\ .$$

We will now consider a discretization of the time interval $[0, T]$ with equally spaces points $\{\gamma^m j\}_{j=0}^J$ where $m \in \mathbb{N}$ and $J = \lfloor \gamma^{-m} T \rfloor$. The value of m will be chosen later. For any $0 \le j < J$ we have that

$$\mathbb{E}\left[|Z_\gamma(\gamma^m j, x) - Z_\gamma(\gamma^m (j+1), x)|^p\right]^{\frac{1}{p}} \le C\gamma^{m\frac{a}{4}} + C\gamma^{\frac{4}{3}} \lesssim (\gamma^m)^{\frac{a}{4}}\ .$$

for m large enough. Let now Z_γ being the approximation of Z_γ obtained interpolating linearly $Z_\gamma(t)$ between the times in $\{\gamma^m j\}_{j=0}^J$. For Z_γ we have that for all $0 < a < \frac{1}{2}$ and $0 \le s \le t \le T$

$$\mathbb{E}\left[|\mathsf{Z}_\gamma(t,x) - \mathsf{Z}_\gamma(s,y)|^p\right]^{\frac{1}{p}} \le C|t-s|^{\frac{a}{4}} + C|x-y|^a\ .$$

and Kolmogorov's compactness criterion implies that the statement of the theorem holds true for the approximation Z_γ and moreover that the limit process belongs to $\mathcal{H}^{\frac{1}{4}-}$. We are now going to show that, for every $\kappa > 0$ and every $p > 1$ there exists a constant C such that

$$\mathbb{E}\left[\sup_{0 \le t \le T} \sup_{x \in \mathbb{T}} |\mathsf{Z}_\gamma(t,x) - Z_\gamma(t,x)|^p\right] \le C\gamma^{p\left(\frac{5}{3}-\kappa\right)}\ . \tag{4.49}$$

At a multiplicative cost of $\log(\epsilon^{-1})$, we can replace the supremum on $\mathbb{T}$ with a supremum over Λ_ε. The cost can be easily absorbed in the choice of $\kappa > 0$.
From the definition of Z_γ, we have that

$$\sup_{0 \le t \le T} |\mathsf{Z}_\gamma(t,x) - Z_\gamma(t,x)| \le \sup_{\substack{j=0,\dots,J \\ j\gamma^m \le t \le (j+1)\gamma^m \wedge T}} 2|Z_\gamma(t,x) - Z_\gamma(\gamma^m j, x)|$$

and therefore, replacing the supremum over $j = 0, \dots, J$ with the summation, we see that the quantity inside the expectation in (4.49) is bounded by

$$\sup_{0 \le t \le T} \|\mathsf{Z}_\gamma(t) - Z_\gamma(t)\|_{L^\infty(\Lambda_\varepsilon)}^p \le \sum_{j=0}^J \sup_{j\gamma^m \le t \le (j+1)\gamma^m \wedge T} 2^p \left\|Z_\gamma(t) - Z_\gamma(\gamma^m j)\right\|_{L^\infty(\Lambda_\varepsilon)}^p \tag{4.50}$$

and we will estimate the right-hand-side of the above equation with (4.43). For $x \in \Lambda_\varepsilon$ and

$t \in [\gamma^m j, \gamma^m (j+1)]$

$$Z_\gamma(t,x) - Z_\gamma(\gamma^m j, x) = \int_{\gamma^m j}^{t} \Delta_\epsilon \Delta_\gamma Z_\gamma(r,x) dr + M_\gamma(t,x) - M_\gamma(\gamma^m j, x) \, .$$

By definition of the discrete Laplacian we have that

$$\|\Delta_\epsilon \Delta_\gamma Z_\gamma(r)\|_{L^\infty(\Lambda_\varepsilon)} \leq 4\epsilon^{-4}\gamma^2 \, \|Z_\gamma(r)\|_{L^\infty(\Lambda_\varepsilon)}$$

and an application of the Hölder's inequality implies that

$$\begin{aligned} \mathbb{E}\left[\sup_{\gamma^m j \leq t \leq \gamma^m(j+1)} \|Z_\gamma(t) - Z_\gamma(\gamma^m j)\|^p_{L^\infty(\Lambda_\varepsilon)}\right] & \\ \leq 4^p \epsilon^{-4p} \gamma^{2p} \gamma^{m(p-1)} \int_{\gamma^m j}^{\gamma^m (j+1)} & \mathbb{E}\left[\|Z_\gamma(r)\|^p_{L^\infty(\Lambda_\varepsilon)}\right] dr \\ & + \mathbb{E}\left[\sup_{\gamma^m j \leq t \leq \gamma^m(j+1)} \|M_\gamma(t) - M_\gamma(\gamma^m j)\|^p_{L^\infty(\Lambda_\varepsilon)}\right] \, . \end{aligned}$$

bounding the supremum with the sum over $x \in \Lambda_\varepsilon$ we obtain that the above quantity is bounded by

$$\begin{aligned} \sum_{x \in \Lambda_\varepsilon} \epsilon^{-4p} \gamma^{(2+m)p} \sup_{\gamma^m j \leq t \leq \gamma^m(j+1)} \mathbb{E}\left[|Z_\gamma(s,x)|^p\right] & \\ + \sum_{x \in \Lambda_\varepsilon} \mathbb{E}\left[\sup_{\gamma^m j \leq t \leq \gamma^m(j+1)} |M_\gamma(t,x) - M_\gamma(\gamma^m j, x)|^p\right] & \quad (4.51) \end{aligned}$$

The first expectation in (4.51) can be bounded by (4.48), while to estimate the last martingale we can us the Burkholder-Davis-Gundy inequality for martingales with jumps. Let Δ_r be the operator that associates to a function $Y : [0,T] \to \mathbb{R}$ its jumps $\Delta_r Y_r := Y_r - Y_{r^-}$. In particular $\Delta_r Y_r$ is non zero only when Y has a jump. If $\{Y_s\}_{0 \leq s \leq t}$ is a real right continuous martingale with jumps then (see [MW17a, Lemma C.1])

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |Y_s|^p\right] \lesssim \mathbb{E}\left[\langle Y \rangle_t^{\frac{p}{2}}\right] + \mathbb{E}\left[\sup_{0 \leq r \leq t} |\Delta_r Y_r|^p\right]$$

To apply the above inequality we need to estimate the quadratic variation and the size of the jumps of the martingale M_γ. The quadratic variation is given in (4.19)

$$\langle M_\gamma(\cdot, x) - M_\gamma(s,x) \rangle_t \lesssim |t-s| \, \|\nabla_\epsilon K_\gamma\|^2_{L^2(\Lambda_\varepsilon)} \lesssim |t-s| \gamma^{-1}$$

for $s \leq t$. We are now going to estimate the jumps $\Delta_r M_\gamma(r,x)$. Recall that M_γ satisfies (4.24) and therefore its jumps coincide with the jumps of X_γ. Each jump is triggered by the exchange of the magnetization value of two neighbouring spins at lattice points, for instance u and $u+1$ in the discrete lattice Λ_N. When the exchange takes place, the value of $X_\gamma(s,x)$ changes less than $\delta^{-1}|\kappa_\gamma(\epsilon^{-1}x-u)-\kappa_\gamma(\epsilon^{-1}x-u-1)| \lesssim \gamma^{\frac{5}{3}}$ and so the value of $M_\gamma(s,x)$. The Burkholder-Davis-Gundy inequality implies that there exists a $C=C(p)$ such that

$$\mathbb{E}\left[\sup_{\gamma^m j \leq t \leq \gamma^m(j+1)} |M_\gamma(t,x)-M_\gamma(\gamma^m j,x)|^p\right] \leq C\gamma^{\frac{pm}{2}}\gamma^{-\frac{p}{2}}+C\gamma^{\frac{5p}{3}} .$$

Therefore (4.51) is bounded by

$$C\epsilon^{-1}\epsilon^{-4p}\gamma^{(2+m)p}+C\epsilon^{-1}\gamma^{\frac{pm}{2}}\gamma^{-\frac{p}{2}}+C\epsilon^{-1}\gamma^{\frac{5p}{3}} .$$

If we set m such that

$$-\frac{16}{3}+2+m>\frac{5}{3}, \qquad \frac{m}{2}-\frac{1}{2}>\frac{5}{3}$$

and perform the summation for $j=0,\dots,J$ in (4.51) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\sup_{x\in\mathbb{T}}|\mathsf{Z}_\gamma(t,x)-Z_\gamma(t,x)|^p\right] \lesssim \gamma^{-m}T\epsilon^{-1}\gamma^{\frac{5p}{3}}$$

and therefore (4.50) is proven for p large enough. The result for every $p\geq 1$ follows by the monotonicity of the L^p norms. □

We are now going to prove the bounds used in the previous theorem.

Lemma 4.4.2 *For all $p\geq 2$ and $0<a<\frac{1}{2}$, there exists $C=C(p,a)$ such that for $x,x'\in\mathbb{T}$ and $0\leq r,s\leq t$*

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|R_{\gamma,t}(s,x)|^p\right]\leq C \tag{4.52}$$

$$\mathbb{E}\left[\sup_{0\leq r\leq t}|R_{\gamma,t}(r,x)-R_{\gamma,s}(r\wedge s,x)|^p\right]\leq C|t-s|^{\frac{a}{4}p}+C\gamma^{\frac{4}{3}p} \tag{4.53}$$

$$\mathbb{E}\left[\sup_{s\leq r\leq t}|R_{\gamma,t}(r,x)-R_{\gamma,t}(s,x)|^p\right]\leq C|t-s|^{\frac{a}{4}p}+C\gamma^{\frac{4}{3}p} \tag{4.54}$$

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|R_{\gamma,t}(s,x)-R_{\gamma,t}(s,x')\right|^p\right]\leq C|x-x'|^{ap} \tag{4.55}$$

Proof. It is easy to show that the above inequalities are valid for $p=2$ from the computation of the quadratic variation (4.47) of the process $R_{\gamma,t}$. Indeed for $x \in \mathbb{T}$, from the definition of the operator Ext,

$$\mathsf{Ext}(R_{\gamma,t}(s))(x) = \frac{1}{2} \sum_{\omega \in \Lambda_N} e^{\pi i x \omega} \hat{R}_{\gamma,t}(s,\omega) = \epsilon \sum_{z \in \Lambda_\varepsilon} \varphi(z) R_{\gamma,t}(s,z)$$

for a certain $\varphi : \Lambda_\varepsilon \to \mathbb{R}$ with $|\hat{\varphi}(\omega)| = 1$. Hence, for any $x \in \mathbb{T}$ and $0 \leq s \leq t$,

$$\langle R_{\gamma,t}(\cdot,x)\rangle_s \lesssim \sum_{\omega \in \Lambda_N \setminus \{0\}} \frac{|\hat{K}_\gamma(\omega)|^2}{\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))} < \infty \, .$$

Moreover, from

$$R_{\gamma,t}(r,x) - R_{\gamma,s}(r \wedge s, x) = \int_{[0,r)} \left(P^{K,\gamma}_{t-r'} - 1_{\{r'<s\}} P^{K,\gamma}_{s-r'} \right) dM_\gamma(r',x)$$

we see that

$$\begin{aligned} &\langle R_{\gamma,t}(\cdot,x) - R_{\gamma,s}(\cdot \wedge s, x)\rangle_r \\ &\qquad \lesssim \int_{r\wedge s}^{r} \epsilon \sum_{z \in \Lambda_\varepsilon} \left(P^{K,\gamma}_{t-r'} \nabla_\epsilon K_\gamma \right)^2 (x-z) dr' \\ &\qquad\quad + \int_0^{r \wedge s} \epsilon \sum_{z \in \Lambda_\varepsilon} \left(P^{K,\gamma}_{t-r'} \nabla_\epsilon K_\gamma - P^{K,\gamma}_{s-r'} \nabla_\epsilon K_\gamma \right)^2 (x-z) dr' \, . \end{aligned}$$

The same computation shows that, for $s \leq r \leq t$

$$\langle R_{\gamma,t}(\cdot,x) - R_{\gamma,t}(s,x)\rangle_r \lesssim \int_s^r \left(P^{K,\gamma}_{t-r'} \nabla_\epsilon K_\gamma(x-z) \right)^2 dr' \, .$$

In a similar way we have that for $x, x' \in \mathbb{T}$ and $0 \leq s \leq t$

$$\begin{aligned} &\big\langle R_{\gamma,t}(\cdot,x) - R_{\gamma,t}(\cdot,x')\big\rangle_s \\ &\qquad \lesssim \int_0^s \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x-z) - P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x'-z) \right)^2 dr \end{aligned}$$

In the appendix it is proved that for every $0 < a < \frac{1}{2}$ there exists $C = C(a)$ such that for all

$0 \leq s' \leq s \leq t$ and every $x, x' \in \Lambda_\varepsilon$ we have

$$\int_{s'}^{s} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(z) \right)^2 dr \leq C(a)|s-s'|^{\frac{a}{2}}$$

$$\int_{0}^{s} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(z) - P^{K,\gamma}_{s-r} \nabla_\epsilon K_\gamma(z) \right)^2 dr \leq C(a)|t-s|^{\frac{a}{2}}$$

$$\int_{0}^{t} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x-z) - P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x'-z) \right)^2 dr \leq C(a)|x-x'|^{2a}$$

and it is clear that this estimates are valid also for $x, x' \in \mathbb{T}$. This is sufficient to prove the statement of the lemma for $p = 2$.

For larger $p > 2$ we will exploit the fact that $R_{\gamma,t}(s,x)$ is a martingale in $0 \leq s \leq t$ to apply the Burkholder-Davis-Gundy inequality. Therefore in order to prove the lemma is sufficient to bound the size of the jumps of the processes. To do so we will use the representation (4.46). If a jump occurs at macroscopic time $0 \leq s \leq t$ and microscopic point $u \in \Lambda_N$, then $R_{\gamma,t}(s,x)$ has a jump of size

$$\begin{aligned}
|\Delta_s R_{\gamma,t}(s,x)| &= 2\delta^{-1} \left| \sum_{z \in \Lambda_N} \epsilon P^{K,\gamma}_{t-s}(x - \epsilon z) \left(\kappa_\gamma(u-z) - \kappa_\gamma(u+1-z) \right) \right| \\
&\leq 2\delta^{-1}\epsilon^2 \left\| P^{K,\gamma}_{t-s} \nabla_\epsilon K_\gamma \right\|_{L^\infty(\Lambda_\varepsilon)} \lesssim \delta^{-1}\epsilon^2 \sum_{\omega \in \Lambda_N} |\omega||\hat{K}_\gamma(\omega)| \lesssim \gamma^{2-\frac{1}{3}} \log(\gamma^{-1})
\end{aligned}$$

uniformly in s, t and $x \in \Lambda_\varepsilon$. In the last inequality we used the bounds in Proposition B.0.1. It is easy to see that the bound holds true also for $x \in \mathbb{T}$. Hence the first inequality (4.52) is proven. From the same bound on the jump, (4.53) follows.

In a similar way, for $x, x' \in \mathbb{T}$, using the inequality $|e^{\pi i x \omega} - e^{\pi i x' \omega}| \leq C|x-x'|^\lambda |\omega|^\lambda$ for all $0 \leq \lambda \leq 1$ we have

$$\begin{aligned}
&|\Delta_s \left(R_{\gamma,t}(s,x) - R_{\gamma,t}(s,x') \right)| \\
&= 2\delta^{-1} \left| \sum_{z \in \Lambda_N} \epsilon \left(P^{K,\gamma}_{t-s}(x - \epsilon z) - P^{K,\gamma}_{t-s}(x' - \epsilon z) \right) \left(\kappa_\gamma(u-z) - \kappa_\gamma(u+1-z) \right) \right| \\
&\leq 2\delta^{-1}\epsilon^2 \left| P^{K,\gamma}_{t-s} \nabla_\epsilon K_\gamma(u-x) - P^{K,\gamma}_{t-s} \nabla_\epsilon K_\gamma(u-x') \right| \\
&\lesssim \delta^{-1}\epsilon^2 |x-x'|^\lambda \sum_{\omega \in \Lambda_N} |\omega|^{1+\lambda} |\hat{K}_\gamma(\omega)| \\
&\lesssim |x-x'|^\lambda \left(\delta^{-1}\epsilon^{-\lambda}\gamma^{2+\lambda} + \delta^{-1}\epsilon^2 (\epsilon^{-1}\gamma)^\lambda \right) \leq \gamma^{2-\frac{1}{3}} \epsilon^{-\lambda} |x-x'|^\lambda
\end{aligned}$$

and (4.55) follows from the choice $\lambda = a$. □

By Theorem 4.4.1, there exists a subsequence converging in law to a process $\underline{Z}$. By the consideration over the size of the jumps of Z_γ, one can see that $\underline{Z}$ is supported in $\mathcal{C}([0,T];\mathcal{C}(\mathbb{T}))$. In the next theorem we will prove the characterization of the law of $\underline{Z}$.

Theorem 4.4.3 (Characterization of the law of Z_γ) *As $\gamma \to 0$, the processes Z_γ converge in law to the solution of* (4.44).

Proof. The proof is based on the martingale characterization of the law of (4.44). Therefore, it is sufficient to show that, for all $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ the following quantities are local martingales

$$\mathcal{M}(\varphi,t) := \langle \underline{Z}(t), \varphi\rangle - \int_0^t \left\langle \underline{Z}(s), -\Delta^2\varphi \right\rangle \, ds$$
$$\mathcal{N}(\varphi,t) := (\mathcal{M}(\varphi,t))^2 - 2t \left\| \nabla\varphi \right\|^2_{L^2(\mathbb{T})}$$

where $\langle \cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{T})$. Let $F : \mathcal{D}([0,T];\mathcal{C}) \to \mathbb{R}$ be a continuous and bounded function measurable with respect to the σ-algebra generated by $\mathcal{D}([0,s];\mathcal{C})$ for $0 \le s \le T$. We will now show that

$$\mathbb{E}\left[\left(\langle \underline{Z}(t),\varphi\rangle - \langle \underline{Z}(s),\varphi\rangle + \int_s^t \left\langle \underline{Z}(s), \Delta^2\varphi\right\rangle \, ds\right) F(\underline{Z})\right] = 0$$

We will consider Z_γ as an element in $\mathcal{C}(\mathbb{T})$, extended with the operator Ext. By (4.43) we have that

$$\mathbb{E}\left[\left(\langle Z_\gamma(t),\varphi\rangle - \langle Z_\gamma(s),\varphi\rangle + \int_s^t \langle Z_\gamma(s), \Delta_\gamma\Delta_\epsilon\varphi\rangle \, ds\right) F(Z_\gamma)\right] = 0\,,$$

and in order to pass to the limit $\gamma \to 0$, we need to replace

$$\begin{aligned}&\left|\langle Z_\gamma(s), \Delta_\gamma\Delta_\epsilon\varphi\rangle - \left\langle Z_\gamma(s), \Delta^2\varphi\right\rangle\right| \\ &\qquad \le \left\|Z_\gamma(s)\right\|_{L^\infty(\mathbb{T})} \left(\left\|\Delta_\gamma\Delta_\epsilon\varphi - \Delta\Delta_\epsilon\varphi\right\|_{L^1(\mathbb{T})} + \left\|\Delta\Delta_\epsilon\varphi - \Delta\Delta\varphi\right\|_{L^1(\mathbb{T})}\right)\end{aligned}$$

By the fact that $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ we have

$$\left\|\Delta_\gamma\Delta_\epsilon\varphi - \Delta\Delta_\epsilon\varphi\right\|_{L^1(\mathbb{T})} \lesssim \gamma\,, \qquad \left\|\Delta_\epsilon\Delta\varphi - \Delta\Delta\varphi\right\|_{L^1(\mathbb{T})} \lesssim \epsilon$$

Moreover $\mathbb{E}\left\|Z_\gamma(s)\right\|_{L^\infty(\mathbb{T})} < \infty$ by Theorem 4.4.1, hence

$$\left|\mathbb{E}\left[\left(\langle Z_\gamma(t),\varphi\rangle - \langle Z_\gamma(s),\varphi\rangle + \int_s^t \left\langle Z_\gamma(s), \Delta^2\varphi\right\rangle \, ds\right) F(Z_\gamma)\right]\right| \lesssim \gamma$$

We will need another approximation of the above quantity. Let $R > 0$ and consider a bounded function $\iota_R : \mathbb{R} \to [-R-1, R+1]$ such that

$$\iota_R(x) = \begin{cases} x & \text{if } x \in [-R, R] \\ R+1 & \text{if } x \geq R+2 \\ -R-1 & \text{if } x \leq -R-2 \end{cases}$$

and with $\|\iota_R'\|_\infty \leq 1$. Therefore

$$\left|\mathbb{E}\left[\left(\iota_R(\langle Z_\gamma(t), \varphi\rangle) - \iota_R(\langle Z_\gamma(s), \varphi\rangle) - \int_s^t \iota_R(\langle Z_\gamma(s), \Delta^2\varphi\rangle)\, ds\right) F(Z_\gamma)\right]\right|$$
$$\leq C(\varphi, F)\gamma + C(\varphi, F)\mathbb{E}\left[\|Z_\gamma\|_{L^\infty([0,T]\times\mathbb{T})} 1_{\{\sup_{0\leq s\leq T} |\langle Z_\gamma(s), \Delta^2\varphi\rangle| > R\}}\right] .$$

By Theorem 4.4.1, the last term is vanishing as $R \to \infty$ uniformly in γ. Then, for any fixed φ, F as above and positive R, the function

$$z \mapsto \left(\iota_R(\langle z(t), \varphi\rangle) - \iota_R(\langle z(s), \varphi\rangle) - \int_s^t \iota_R(\langle z(s), \Delta^2\varphi\rangle)\, ds\right) F(z)$$

is bounded and continuous in z with respect to the topology of $\mathcal{D}([0,T]; \mathcal{C}(\mathbb{T}))$. Therefore taking the limit $\gamma \to 0$ and using the fact that $Z_\gamma \xrightarrow{\mathcal{L}} \underline{Z}$

$$\left|\mathbb{E}\left[\left(\iota_R(\langle \underline{Z}(t), \varphi\rangle) - \iota_R(\langle \underline{Z}(s), \varphi\rangle) - \int_s^t \iota_R(\langle \underline{Z}(s), \Delta^2\varphi\rangle)\, ds\right) F(\underline{Z})\right]\right|$$
$$\leq C(\varphi, F) o_R(1) .$$

By Portmanteau's theorem the probability

$$\mathbb{P}\left(\sup_{0\leq s\leq T} \|\underline{Z}(s)\|_{L^\infty(\mathbb{T})} > R\right) \to 0$$

as R goes to infinity and this completes the result for the first martingale.
In order to prove the statement for the second martingale $\mathcal{M}^2$ we will need Conjecture 4.2.6. Let the functions F and φ being defined as above: we need to show that

$$\mathbb{E}\left[\left(\mathcal{M}^2(\varphi, t) - \mathcal{M}^2(\varphi, s) - 2(t-s)\|\nabla\varphi\|^2_{L^2(\mathbb{T})}\right) F(\underline{Z})\right] .$$

Define

$$M_\gamma(\varphi, t) = \sum_{x\in\Lambda_\varepsilon} \epsilon M_\gamma(x,t)\varphi(x) ,$$

from (4.19) we see that the quadratic variation $\langle M_\gamma(\varphi,\cdot)\rangle_t$ is given by

$$4\int_0^t \epsilon \sum_{z\in\Lambda_N} c_\gamma^K(z,z+1,\sigma(\alpha^{-1}r^-))1_{\{\sigma_z(\alpha^{-1}r^-)\neq\sigma_{z+1}(\alpha^{-1}r^-)\}}(\nabla_\epsilon K_\gamma *_\epsilon \varphi(\epsilon z))^2\, dr$$

and using the fact that

$$2c_\gamma^K(z,z+1,\sigma(s))1_{\{\sigma_z(s)\neq\sigma_{z+1}(s)\}} = (1+\mathcal{O}(\gamma))\,(1-\sigma_z(s)\sigma_{z+1}(s))$$

we have that

$$\begin{aligned}&\left|\langle M_\gamma(\varphi,\cdot)\rangle_t - 2t\,\|\nabla_\epsilon K_\gamma *_\epsilon \varphi\|^2_{L^2(\Lambda_\varepsilon)}\right| \leq \\ &\qquad C(\varphi)\gamma t + 2\left|\int_0^t \epsilon \sum_{z\in\Lambda_N} \sigma_z(\alpha^{-1}s^-)\sigma_{z+1}(\alpha^{-1}s^-)(\nabla_\epsilon K_\gamma *_\epsilon \varphi(\epsilon z))^2\, ds\right| .\end{aligned}$$

In order to complete the proof we need to show that for all $t>0$, as $\gamma\to 0$

$$\begin{aligned}&\mathbb{E}\left[M_\gamma^2(\varphi,t)F(Z_\gamma)\right] \to \mathbb{E}\left[\mathcal{M}^2(\varphi,t)F(\underline{Z})\right]\\ &\|\nabla_\epsilon K_\gamma *_\epsilon \varphi\|^2_{L^2(\Lambda_\varepsilon)} \to \|\nabla\varphi\|^2_{L^2(\Lambda_\varepsilon)}\end{aligned}$$

and that

$$\mathbb{E}\left[\left|\int_0^t \epsilon \sum_{z\in\Lambda_N} \sigma_z(\alpha^{-1}s^-)\sigma_{z+1}(\alpha^{-1}s^-)(\nabla_\epsilon K_\gamma *_\epsilon \varphi(\epsilon z))^2\, ds\right|\right] \to 0 \tag{4.56}$$

The first set of limits follows from (4.43), the tightness of the process Z_γ and the smoothness of φ. Let

$$U_\gamma^\varphi(s,\sigma) := \epsilon \sum_{z\in\Lambda_N} \left\{\sigma_z\sigma_{z+1}(\alpha^{-1}s^-) - h_\gamma^2(z,\alpha^{-1}s^-)\right\}(\nabla_\epsilon K_\gamma *_\epsilon \varphi(\epsilon z))^2$$

We claim that uniformly in the initial conditions, for all fixed $b>0$

$$\limsup_{\gamma\to 0} \mathbb{P}\left(\left|t^{-1}\int_0^t U_\gamma^\varphi(s,\sigma)\, ds\right| > b\right) = 0\,. \tag{4.57}$$

The limit in (4.57) is a consequence of the superexponential estimate (see [KOV89]), tailored for the time scale of the Kawasaki process. The time scale of the Kawasaki process and the mesoscopic size of the blocks, guarantee a fast mixing of the magnetization, much better than in the case of the simmetric simple exclusion process, that forces (4.57) to vanish.

Assuming (4.57), we have that (4.56) is bounded by

$$bt + t\mathbb{P}\left(\left|t^{-1}\int_0^t U_\gamma^\varphi(s,\sigma)\,ds\right| > b\right) \\ + \mathbb{E}\left[\int_0^t \epsilon \sum_{z\in\Lambda_\varepsilon} \left\{\delta^2 X_\gamma^2(z,s)\wedge 1\right\}(\nabla_\epsilon K_\gamma *_\epsilon \varphi(z))^2\,ds\right]$$

and therefore to complete the proposition it is sufficient to apply some mild control over the field X_γ. We recall that $X_\gamma = Z_\gamma + V_\gamma$ where V_γ is the remainder process. From (4.48) we have that

$$\delta^2\mathbb{E}\left[\int_0^t \epsilon \sum_{z\in\Lambda_\varepsilon} Z_\gamma^2(z,s^-)(\nabla_\epsilon K_\gamma *_\epsilon \varphi(z))^2\,ds\right] \lesssim \delta^2 t\left\|\nabla_\epsilon K_\gamma *_\epsilon \varphi\right\|_{L^2(\Lambda_\varepsilon)}^2$$

with $\delta = \gamma^{\frac{2}{3}}$ and therefore the proposition is complete if we can show that

$$\lim_{\gamma\to 0}\mathbb{E}\left[\int_0^t \left\{\delta^2\left\|V_\gamma(s)\right\|_{L^2(\Lambda_\varepsilon)}^2 \wedge 1\right\}\,ds\right] \to 0\,.$$

But this is a consequence of the fact that, for any constants $b,\lambda > 0$ we have that

$$\mathbb{E}\left[\delta^2\left\|V_\gamma(s)\right\|_{L^2(\Lambda_\varepsilon)}^2 \wedge 1\right] \le \mathbb{P}\left(\left\|V_\gamma(s)\right\|_{L^2(\Lambda_\varepsilon)} > b\gamma^{-\lambda}\right) + b^2\gamma^{-2\lambda}\delta^2$$

and, for a sufficiently small λ, it is sufficient to apply Corollary 4.5.5 proved in the next section. □

4.5 Convergence for the nonlinear process

We now establish the convergence in law to the mild solution of the one-dimensional stochastic Cahn-Hilliard equation. We shall now define an approximation $\bar{X}_\gamma$ to X_γ, obtained ignoring the contribution of $\mathcal{U}_\gamma(x,t)$.

Recall the definition of V_γ in (4.26), and define $\bar{V}_\gamma(x,t)$ to be the solution on $[0,T]\times\Lambda_\varepsilon$ to the following

$$\begin{cases} \partial_t\bar{V}_\gamma(x,t) &= -\Delta_\epsilon\left\{\Delta_\gamma\bar{V}_\gamma(x,t) + K_\gamma *_\epsilon F_\gamma\left(Z_\gamma(x,t)+\bar{V}_\gamma(x,t)\right)\right\} \\ \bar{V}_\gamma(x,0) &= X_\gamma^0(x) \end{cases} \tag{4.58}$$

We will think of F_γ as being the polynomial of degree three with negative leading coefficient

$$F_\gamma(x) = A_\gamma x - \frac{B_\gamma}{3}x^3 \tag{4.59}$$

where $A_\gamma = A + \mathcal{O}(\gamma^{\frac{1}{3}})$, $B_\gamma = 1 + \mathcal{O}(\gamma^{\frac{1}{3}})$ and $\mathfrak{g}_\gamma$ is given in (4.22).The form of (4.58) is very similar to (4.39). Indeed it is possible to prove that

Proposition 4.5.1 *The process $\bar{V}_\gamma$ defined in* (4.58) *satisfies*

$$\sup_{0\le t\le T} \left\|\bar{V}_\gamma(t)\right\|^2_{L^2(\Lambda_\varepsilon)} \le C(T)\left(1 + \left\|X^0_\gamma\right\|^2_{L^2(\Lambda_\varepsilon)}\right)\left(1 + \sup_{0\le s\le T}\|Z_\gamma(s)\|^6_{L^\infty(\Lambda_\varepsilon)}\right)$$

The proof of Proposition 4.5.1 follows the strategy outlined in the proof of Proposition 4.3.3, which relies essentially on the possibility of performing summations by parts, the presence of nonpositive differential operators and the Young's inequality, which are tools available also for the discrete PDE (4.58). In the proof it might be convenient to take into consideration the following inequality, that easily follows from Plancherel's theorem and Proposition B.0.1

$$\left\langle \Delta_\epsilon \bar{V}_\gamma, K_\gamma *_\epsilon \Delta_\epsilon \bar{V}_\gamma \right\rangle_{\Lambda_\varepsilon} \le \left\langle \nabla_\epsilon \bar{V}_\gamma, (-\Delta_\gamma)\, \nabla_\epsilon \bar{V}_\gamma \right\rangle_{\Lambda_\varepsilon} .$$

Define for $(t,x) \in [0,T] \times \Lambda_\varepsilon$ the approximation $\bar{X}_\gamma \stackrel{\text{def}}{=} Z_\gamma + \bar{V}_\gamma$.
From the mild form of (4.58) it is possible to see that $\bar{X}_\gamma$ satisfies the following

$$\bar{X}_\gamma(x,t) = P^{K,\gamma}_t X^0_\gamma(x) + \int_0^t P^{K,\gamma}_{t-s}(-\Delta_\epsilon) K_\gamma *_\epsilon F_\gamma\left(\bar{X}_\gamma(x,s)\right) ds + Z_\gamma(x,t) \tag{4.60}$$

for $(t,x) \in [0,T] \times \Lambda_\varepsilon$.

For the approximation $\bar{X}_\gamma(x,t)$, using (4.60) and Theorem 4.4.3, one is able to prove the convergence in law to the mild solution of (4.30). To keep the notations light, we omitted from the following statement the fact that all the processes are extended to whole torus $\mathbb{T}$, via the function Ext.

Theorem 4.5.2 *Let X^0_γ be a deterministic sequence converging to X^0 in the sense of Assumption 4.2.5, and let $\bar{X}_\gamma$ satisfy* (4.60).
Then, for any $T > 0$, the approximation $\bar{X}_\gamma$ converges in law to the solution of (4.8) *in* $\mathcal{D}([0,T]; L^\infty(\mathbb{T}))$.

Before the proof of Theorem 4.5.2, we are going to discuss how we plan to use it to prove the convergence for the law of the original process X_γ as stated in Theorem 4.2.7. Recall that, from (4.60) and (4.27), $X_\gamma = \bar{X}_\gamma + V_\gamma - \bar{V}_\gamma$. It is not true in general, that if two sequences of random variables $Q^{(n)}_1, Q^{(n)}_2$ are convergent in distribution $Q^{(n)}_1 \xrightarrow{\mathscr{L}} Q_1$ and $Q^{(n)}_2 \xrightarrow{\mathscr{L}} Q_2$, their sum is also converging in distribution. However, if for instance Q_2 is constant, then $Q^{(n)}_2 \xrightarrow{\mathbb{P}} Q_2$ and $Q^{(n)}_1 + Q^{(n)}_2 \xrightarrow{\mathscr{L}} Q_1 + Q_2$ holds true. This means that

Theorem 4.2.7 is proved if we can show the next proposition.

Proposition 4.5.3 (Conditioned on Conjecture 4.2.6) *Assume the statement of Conjecture 4.2.6.*
Recall the definitions of the processes V_γ and $\bar{V}_\gamma$ defined respectively in (4.27) *and* (4.58). *As $\gamma \to 0$, the difference $V_\gamma - \bar{V}_\gamma$ converges in distribution in $\mathcal{C}([0,T], L^\infty(\mathbb{T}))$ to the process identically equal to* 0.

The above theorem can be formulated more directly in terms of convergence in probability for $V_\gamma - \bar{V}_\gamma$ in $\mathcal{C}([0,T], L^\infty(\mathbb{T}))$, namely

Proposition 4.5.4 (Conditioned on Conjecture 4.2.6) *Assume the statement of Conjecture 4.2.6.*
For all $b > 0$

$$\limsup_{\gamma \to 0} \mathbb{P}\left(\sup_{0 \le t \le T} \left\| V_\gamma(t) - \bar{V}_\gamma(t) \right\|_{L^\infty(\mathbb{T})} > b \right) = 0$$

It is easy to see that Proposition 4.5.4 is proven if, for all $b > 0$

$$\limsup_{\gamma \to 0} \mathbb{P}\left(\sup_{0 \le t \le T; x \in \Lambda_\varepsilon} \left| \int_0^t P^{K,\gamma}_{t-s} \mathcal{U}_\gamma(x,s)\, ds \right| > b \right) = 0\,.$$

We are now going to prove Theorem 4.5.2. The proof is based on [MW17a, Sec. 7].

Proof of Theorem 4.5.2. Recall that the solution of the stochatic Cahn-Hilliard equation (4.8) can be decomposed into

$$X(t) = Z(t) + \mathcal{S}^K_T(Z, X^0)(t)$$

where $\mathcal{S}^K_T$ is defined in Proposition 4.3.3. By Assumption 4.2.5, the sequence X^0_γ converges to a deterministic limit and $\mathsf{Ext} Z_\gamma \xrightarrow{\mathscr{L}} Z$ in $\mathcal{D}([0,T]; L^\infty(\mathbb{T}))$ and the couple $(\mathsf{Ext} Z_\gamma, \mathsf{Ext} X^0_\gamma)$ jointly converges in law to (Z, X^0). Let us define $\overline{\overline{V}}_\gamma : \mathbb{T} \to \mathbb{R}$ as

$$\overline{\overline{V}}_\gamma \stackrel{\text{def}}{=} \mathcal{S}^K_T(\mathsf{Ext} Z_\gamma, \mathsf{Ext} X^0_\gamma) \tag{4.61}$$

and let $\overline{\overline{X}}_\gamma \stackrel{\text{def}}{=} \mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma$. By the Lipschitz continuity of the map $\mathcal{S}^K_T$, and Theorem 4.4.1 we have that

$$\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma \xrightarrow{\mathscr{L}} Z + \mathcal{S}^K_T(Z, X^0)$$

converges in distribution in $\mathcal{D}([0,T]; L^\infty(\mathbb{T}))$. In order to complete the proof, it is sufficient to show that

$$\sup_{0 \le t \le T} \left\| \mathsf{Ext} \bar{V}_\gamma(t) - \overline{\overline{V}}_\gamma(t) \right\|_{L^\infty} \xrightarrow{\mathbb{P}} 0 \tag{4.62}$$

By (4.58), $\bar{V}_\gamma(t)$ can be written in mild form for $x \in \Lambda_\varepsilon$ as

$$\bar{V}_\gamma(x,t) = P_t^{K,\gamma} X_\gamma^0(x) + \int_0^t (-\Delta_\epsilon) P_{t-s}^{K,\gamma} K_\gamma *_\epsilon F_\gamma \left(Z_\gamma + \bar{V}_\gamma\right)(x,s) ds$$

the above relation can be extended to the whole torus via the extension operator Ext: in doing so we will now interpret $P_t^{K,\gamma}$ as a pseudo differential operator. It follows that, for $x \in \mathbb{T}$

$$\mathsf{Ext}\bar{V}_\gamma(x,t) = P_t^{K,\gamma} \mathsf{Ext} X_\gamma^0(x) + \int_0^t (-\Delta_\epsilon) P_{t-s}^{K,\gamma} K_\gamma \star \mathsf{Ext} F_\gamma \left(Z_\gamma + \bar{V}_\gamma\right)(x,s) ds \,.$$

Recall moreover that the process $\overline{\overline{V}}_\gamma$ satisfies

$$\overline{\overline{V}}_\gamma(x,t) = P_t^{K} \mathsf{Ext} X_\gamma^0(x) + \int_0^t (-\Delta) P_{t-s}^{K} \star F\left(\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma\right)(x,s) ds$$

hence the difference satisfies

$$\begin{aligned}
&\mathsf{Ext}\bar{V}_\gamma(x,t) - \overline{\overline{V}}_\gamma(x,t) \\
&= P_t^{K,\gamma} \mathsf{Ext} X_\gamma^0(x) - P_t^{K} \mathsf{Ext} X_\gamma^0(x) && (4.63)\\
&+ \int_0^t \left\{\Delta P_{t-s}^{K} - \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma\right\} \star F\left(\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma\right)(x,s) ds && (4.64)\\
&+ \int_0^t (-\Delta_\epsilon) P_{t-s}^{K,\gamma} K_\gamma \star \left\{\mathsf{Ext} F_\gamma\left(Z_\gamma + \bar{V}_\gamma\right) - F\left(\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma\right)\right\}(x,s) ds && (4.65)
\end{aligned}$$

From (B.8) and the Assumption 4.2.5 it follows that, for $\kappa > 0$ small enough, (4.63) is bounded by

$$\left\|\left(P_t^{K,\gamma} - P_t^{K}\right) \mathsf{Ext} X_\gamma^0\right\|_{L^\infty(\mathbb{T})} \le \sum_{\omega \in \Lambda_N} \left|\hat{P}_t^{K,\gamma}(\omega) - P_t^{K}(\omega)\right| |\hat{X}_\gamma^0(\omega)| \le C(\lambda)\gamma^\lambda \,. \tag{4.66}$$

In order to estimate the quantity in (4.64) we use

$$\begin{aligned}
&\left\|\left\{\Delta P_{t-s}^{K} - \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma\right\} \star F\left(\mathsf{Ext} Z_\gamma(s) + \overline{\overline{V}}_\gamma(s)\right)\right\|_{L^\infty(\mathbb{T})} \\
&\qquad \le \left\|\Delta P_{t-s}^{K} - \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma\right\|_{L^2(\mathbb{T}) \to L^\infty(\mathbb{T})} \left\|F\left(\mathsf{Ext} Z_\gamma(s) + \overline{\overline{V}}_\gamma(s)\right)\right\|_{L^2(\mathbb{T})}
\end{aligned}$$

From (B.7), we have that for any $\kappa > 0$

$$\left\|\Delta P_{t-s}^{K} - \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma\right\|_{L^2(\mathbb{T}) \to L^\infty(\mathbb{T})} \le C(\kappa) t^{-\frac{5+\kappa}{8}} \gamma^{\frac{\kappa}{6}}$$

where we used the scaling $\epsilon = \gamma^{\frac{4}{3}}$. From the fact that the polynomial F has degree 3 we have that

$$\left\| F\left(\mathsf{Ext}Z_\gamma(s) + \overline{\overline{V}}_\gamma(s)\right)\right\|_{L^2(\mathbb{T})} \le \sup_{0\le s\le T} \left\{ \|\mathsf{Ext}Z_\gamma(s)\|^3_{L^6(\mathbb{T})} + \left\|\overline{\overline{V}}_\gamma(s)\right\|^3_{L^6(\mathbb{T})} \right\} .$$

By the above considerations, for $0 < \kappa < 3$, let $p, q > 0$ such that $p^{-1} + q^{-1} = 1$ and $p(5+\kappa)/8 < 1$

$$\begin{aligned}
&\sup_{0\le t\le T} \gamma^{\frac{\kappa}{6}} \int_0^t (t-s)^{-\frac{5+\kappa}{8}} \left\| F\left(\mathsf{Ext}Z_\gamma(s) + \overline{\overline{V}}_\gamma(s)\right)\right\|_{L^2(\mathbb{T})} ds \\
&\quad \le C(\kappa)\gamma^{\frac{\kappa}{6}} \sup_{0\le t\le T} \int_0^t (t-s)^{-\frac{5+\kappa}{8}p} + \|\mathsf{Ext}Z_\gamma(s)\|^{3q}_{L^{6q}(\mathbb{T})} + \left\|\mathsf{Ext}\overline{\overline{V}}_\gamma(s)\right\|^{3q}_{L^{6q}(\mathbb{T})} ds \\
&\qquad \le C\Big(T, \kappa, Z_\gamma, X^0_\gamma\Big)\gamma^{\frac{\kappa}{6}} \qquad (4.67)
\end{aligned}$$

and the last quantity is a polynomial function of $\|\mathsf{Ext}Z_\gamma(s)\|_{L^\infty(\mathbb{T})}$ and $\left\|\mathsf{Ext}X^0_\gamma\right\|_{L^\infty(\mathbb{T})}$ because of (4.42), Theorem 4.4.1 and Assumption 4.2.5.
We consider now (4.65) and we divide it into the sum of

$$\begin{aligned}
&\int_0^t (-\Delta_\epsilon) P^{K,\gamma}_{t-s} K_\gamma \star \left\{ \mathsf{Ext}F_\gamma\left(Z_\gamma + \bar{V}_\gamma\right) - F\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)\right\}(x,s) ds \\
&= \int_0^t (-\Delta_\epsilon) P^{K,\gamma}_{t-s} K_\gamma \star \left\{ F_\gamma\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right) - F\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)\right\}(x,s) ds \qquad (4.68)\\
&+ \int_0^t (-\Delta_\epsilon) P^{K,\gamma}_{t-s} K_\gamma \star \left\{ \mathsf{Ext}F_\gamma\left(Z_\gamma + \bar{V}_\gamma\right) - F_\gamma\left(\mathsf{Ext}Z_\gamma + \mathsf{Ext}\bar{V}_\gamma\right)\right\}(x,s) ds \qquad (4.69)\\
&+ \int_0^t (-\Delta_\epsilon) P^{K,\gamma}_{t-s} K_\gamma \star \left\{ F_\gamma\left(\mathsf{Ext}Z_\gamma + \mathsf{Ext}\bar{V}_\gamma\right) - F_\gamma\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)\right\}(x,s) ds \,. \qquad (4.70)
\end{aligned}$$

From the form of (4.59) we see that

$$|F_\gamma(y) - F(y)| \lesssim \gamma^{\frac{1}{3}}(1 + |y|^3)$$

and therefore, in a similar way, for all $\kappa >$ we have that (4.68)

$$\begin{aligned}
&\left\|(-\Delta_\epsilon) P^{K,\gamma}_{t-s} K_\gamma \star \left\{ F_\gamma\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right) - F\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)\right\}(s)\right\|_{L^\infty(\mathbb{T})} \\
&\le C\gamma^{\frac{1}{3}} \left\|\Delta_\epsilon P^{K,\gamma}_{t-s} K_\gamma\right\|_{L^2\to L^\infty} \left\| F_\gamma\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)(s) - F\left(\mathsf{Ext}Z_\gamma + \overline{\overline{V}}_\gamma\right)(s)\right\|_{L^2(\mathbb{T})} \\
&\qquad \le C(\kappa)\gamma^{\frac{1}{3}}(t-s)^{-\frac{5+\kappa}{8}} \left(1 + \left\|\mathsf{Ext}Z_\gamma(s) + \overline{\overline{V}}_\gamma(s)\right\|^3_{L^6(\mathbb{T})}\right) \qquad (4.71)
\end{aligned}$$

where we used Lemma B.1.4. Integrating (4.71) over time and using the same inequalities of (4.67) yields

$$\int_0^t \left\| \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma \star \left\{ F_\gamma \left(\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma \right) - F \left(\mathsf{Ext} Z_\gamma + \overline{\overline{V}}_\gamma \right) \right\} (s) \right\|_{L^\infty(\mathbb{T})} ds$$
$$\leq C\Big(T, \kappa, q, \sup_{0 \leq s \leq T} \|\mathsf{Ext} Z_\gamma(s)\|_{L^\infty(\mathbb{T})}, \left\|\mathsf{Ext} X_\gamma^0\right\|_{L^\infty(\mathbb{T})} \Big) \gamma^{\frac{1}{3}} \quad (4.72)$$

where the constant depends on its last two arguments polynomially. In order to bound (4.69) we notice that

$$\mathsf{Ext} F_\gamma \left(Z_\gamma + \bar{V}_\gamma \right) - F_\gamma \left(\mathsf{Ext} Z_\gamma + \mathsf{Ext} \bar{V}_\gamma \right) = \frac{B_\gamma}{3} \mathsf{Ext} \bar{X}_\gamma^3 - \frac{B_\gamma}{3} (\mathsf{Ext} \bar{X}_\gamma)^3 \ .$$

Consider now the decomposition into high and low frequency of $Y : \Lambda_\varepsilon \to \mathbb{R}$

$$Y^{\mathrm{low}} = \sum_{|\omega| \leq \epsilon^{-1}/3} \hat{Y}(\omega) \ , \qquad Y^{\mathrm{high}} = \sum_{\epsilon^{-1}/3 < |\omega| \leq \epsilon^{-1}} \hat{Y}(\omega)$$

and apply it to $Y = \bar{X}_\gamma$.

$$\begin{aligned} &\mathsf{Ext} \bar{X}_\gamma^3 - (\mathsf{Ext} \bar{X}_\gamma)^3 \\ &\quad = \left[\mathsf{Ext} (\bar{X}_\gamma^{\mathrm{low}})^3 - (\mathsf{Ext} \bar{X}_\gamma^{\mathrm{low}})^3 \right] \\ &\quad + \mathsf{Ext} \left[3 \bar{X}_\gamma^{\mathrm{high}} (\bar{X}_\gamma^{\mathrm{low}})^2 + 3 \bar{X}_\gamma^{\mathrm{low}} (\bar{X}_\gamma^{\mathrm{high}})^2 + (\bar{X}_\gamma^{\mathrm{high}})^3 \right] \\ &\quad - \left[3 \mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}} (\mathsf{Ext} \bar{X}_\gamma^{\mathrm{low}})^2 + 3 \mathsf{Ext} \bar{X}_\gamma^{\mathrm{low}} (\mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}})^2 + (\mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}})^3 \right] \end{aligned}$$

by the definition of the Ext operator, the first line of the right-hand-side vanishes and therefore

$$\left\| \mathsf{Ext} \bar{X}_\gamma^3 - (\mathsf{Ext} \bar{X}_\gamma)^3 \right\|_{L^1(\mathbb{T})}$$
$$\lesssim \left\| \mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}} \right\|_{L^\infty(\mathbb{T})} \Big(\left\| \mathsf{Ext} \bar{X}_\gamma^{\mathrm{low}} \right\|_{L^2(\mathbb{T})} + \left\| \mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}} \right\|_{L^2(\mathbb{T})} \Big)^2$$

and

$$\left\| \mathsf{Ext} \bar{X}_\gamma^{\mathrm{high}} \right\|_{L^2(\mathbb{T})} \leq \left\| \bar{X}_\gamma \right\|_{L^2(\Lambda_\varepsilon)} \ , \qquad \left\| \mathsf{Ext} \bar{X}_\gamma^{\mathrm{low}} \right\|_{L^2(\mathbb{T})} \leq \left\| \bar{X}_\gamma \right\|_{L^2(\Lambda_\varepsilon)} \ .$$

We can therefore bound 4.69 using Proposition 4.5.1

$$
\begin{aligned}
&C \int_0^t \left\| \Delta_\epsilon P_{t-s}^{K,\gamma} K_\gamma \right\|_{L^\infty(\mathbb{T})} \left\| \mathsf{Ext} \bar{X}_\gamma^{\text{high}}(\cdot, s) \right\|_{L^\infty(\mathbb{T})} \left\| \bar{X}_\gamma(\cdot, s) \right\|_{L^2(\Lambda_\varepsilon)}^2 ds \\
&\quad \lesssim \int_0^t \sum_{\omega \in \Lambda_N} |\omega|^2 \hat{P}_{t-s}^{K,\gamma}(\omega) |\hat{K}_\gamma(\omega)| ds \left\| \mathsf{Ext} \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})} \left\| \bar{X}_\gamma \right\|_{L^\infty([0,T], L^2(\Lambda_\varepsilon))}^2 \\
&\quad \lesssim C\big(\left\| X_\gamma^0 \right\|_{L^\infty}, \sup_{0 \le t \le T} \| Z_\gamma \|_{L^\infty(\Lambda_\varepsilon)} \big) \left\| \mathsf{Ext} \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})} \sum_{\omega \in \Lambda_N \setminus \{0\}} |\omega|^{-2} .
\end{aligned}
$$

In order to control the term $\left\| \mathsf{Ext} \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})}$, we are going to apply to the relation (4.60) the projection in Fourier modes over the frequencies $3^{-1}\epsilon^{-1} < |\omega| \le \epsilon^{-1}$

$$
\begin{aligned}
\bar{X}_\gamma^{\text{high}}(x,t) = \Pi_{3^{-1}\epsilon^{-1}}^{\epsilon^{-1}} P_t^{K,\gamma} X_\gamma^0(x) + \int_0^t \Pi_{3^{-1}\epsilon^{-1}}^{\epsilon^{-1}} P_{t-s}^{K,\gamma} (-\Delta_\epsilon) K_\gamma *_\epsilon F_\gamma\left(\bar{X}_\gamma(x,s) \right) ds \\
+ \Pi_{3^{-1}\epsilon^{-1}}^{\epsilon^{-1}} Z_\gamma(x,t) \ .
\end{aligned}
$$

The supremum over $x \in \Lambda_\varepsilon$ yields

$$
\begin{aligned}
&\left\| \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty(\Lambda_\varepsilon)} \lesssim \| Z_\gamma \|_{L^\infty([0,T]\times\mathbb{T})} \\
&+ \int_0^t \sum_{k=3^{-1}\epsilon^{-1}}^{\epsilon^{-1}} \epsilon^{-2}\gamma^2 e^{-(t-s)\epsilon^{-2}\gamma^2|\omega|^2} \left\| \bar{X}_\gamma^{\text{high}}(\cdot, s) \right\|_{L^\infty(\Lambda_\varepsilon)} \left\| \bar{X}_\gamma(\cdot, s) \right\|_{L^2(\Lambda_\varepsilon)}^2 ds + \epsilon^\lambda C\left(X_\gamma^0 \right)
\end{aligned}
$$

where we used the Assumption 4.2.5. Using Proposition 4.5.1 we are able to bound $\left\| \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)}$ with

$$
\begin{aligned}
\left\| \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)} &\lesssim \left\| Z_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})} + \epsilon \left\| \bar{X}_\gamma \right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)} + \epsilon^\lambda \\
&\lesssim \left\| Z_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})} + \epsilon\delta^{-1} + \epsilon^\lambda
\end{aligned}
$$

where the constant depends on $\| Z_\gamma \|_{L^\infty(\Lambda_\varepsilon)}$, X_γ^0 and we used the fact that $\left\| \bar{X}_\gamma \right\|_{L^\infty(\Lambda_\varepsilon)} \le \delta^{-1}$ deterministically. This shows essentially that the behaviour of $\left\| \bar{X}_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)}$ is controlled by $\left\| Z_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})}$. Hence (4.69) is bounded in norm $L^\infty([0,T]\times\Lambda_\varepsilon)$ by

$$
C\Big(\lambda, \sup_{0 \le s \le T} \| \mathsf{Ext} Z_\gamma(s) \|_{L^\infty(\mathbb{T})}, \left\| \mathsf{Ext} X_\gamma^0 \right\|_{L^\infty(\mathbb{T})} \Big) \left[\left\| Z_\gamma^{\text{high}} \right\|_{L^\infty([0,T]\times\mathbb{T})} + \epsilon\delta^{-1} + \epsilon^\lambda \right]
$$

We will now bound the main term (4.70). Using Lemma B.1.4 and the fact that F_γ is a polynomial of degree 3 we have that (4.70) is bounded by

$$\begin{aligned}
&\int_0^t \left\|(-\Delta_\epsilon)P_{t-s}^{K,\gamma}K_\gamma \star \left\{F_\gamma\left(\mathsf{Ext}\bar{X}_\gamma(s)\right) - F_\gamma\left(\overline{\overline{X}}_\gamma(s)\right)\right\}\right\|_{L^\infty(\mathbb{T})} ds \\
&\quad \le \int_0^t \left\|(-\Delta_\epsilon)P_{t-s}^{K,\gamma}K_\gamma\right\|_{L^1(\mathbb{T})\to L^\infty(\mathbb{T})} \left\|F_\gamma\left(\mathsf{Ext}\bar{X}_\gamma(s)\right) - F_\gamma\left(\overline{\overline{X}}_\gamma(s)\right)\right\|_{L^1(\mathbb{T})} ds \\
&\qquad\qquad \le C(\kappa)\int_0^t (t-s)^{-\frac{3+\kappa}{4}} \left\|\mathsf{Ext}\bar{V}_\gamma(s) - \overline{\overline{V}}_\gamma(s)\right\|_{L^\infty(\mathbb{T})} \\
&\qquad\qquad\qquad \times \left(1 + \left\|\mathsf{Ext}\bar{X}_\gamma(s)\right\|_{L^2(\mathbb{T})}^2 + \left\|\overline{\overline{X}}_\gamma(s)\right\|_{L^2(\mathbb{T})}^2\right) ds\ .
\end{aligned}$$

Recall that $\left\|\mathsf{Ext}\bar{X}_\gamma(s)\right\|_{L^2(\mathbb{T})} = \left\|\bar{X}_\gamma(s)\right\|_{L^2(\Lambda_\varepsilon)}$. Propositions 4.5.1 and the inequality (4.42) guarantee that

$$\sup_{0\le s\le T}\left\|\mathsf{Ext}\bar{X}_\gamma(s)\right\|_{L^2(\mathbb{T})}^2 + \left\|\overline{\overline{X}}_\gamma(s)\right\|_{L^2(\mathbb{T})}^2 \le C\big(Z_\gamma, X_\gamma^0\big)$$

and (4.70) is bounded by

$$C\big(\kappa, Z_\gamma, X_\gamma^0\big)\int_0^t (t-s)^{-\frac{3+\kappa}{4}}\left\|\mathsf{Ext}\bar{V}_\gamma(s) - \overline{\overline{V}}_\gamma(s)\right\|_{L^\infty(\mathbb{T})} ds \tag{4.73}$$

where the constant depends polynomially on $\sup_{0\le s\le T}\left\|\mathsf{Ext}Z_\gamma\right\|_{L^\infty}$ and $\left\|\mathsf{Ext}X_\gamma^0\right\|_{L^\infty}$. Collecting the bounds (4.66),(4.67),(4.72) and (4.73) we obtain that the difference in (4.62) satisfies the Grönwall-type inequality

$$\begin{aligned}
&\left\|\mathsf{Ext}\bar{V}_\gamma(t) - \overline{\overline{V}}_\gamma(t)\right\|_{L^\infty(\mathbb{T})} \\
&\quad \le C\left(\gamma^\lambda + \left\|Z_\gamma^{\text{high}}\right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)} + \int_0^t (t-s)^{-\frac{3+\kappa}{4}}\left\|\mathsf{Ext}\bar{V}_\gamma(s) - \overline{\overline{V}}_\gamma(s)\right\|_{L^\infty(\mathbb{T})} ds\right)
\end{aligned} \tag{4.74}$$

where $C = C\big(\kappa, \lambda, T, Z_\gamma, X_\gamma^0\big)$. From the proof of Theorem 4.4.1 and in particular from (4.47), one can see that the quantity

$$\mathbb{E}\left[\left\|Z_\gamma^{\text{high}}\right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)}^2\right] \le \epsilon^{-1}\mathbb{E}\left[\sup_{0\le t\le T}\left\|Z_\gamma^{\text{high}}(\cdot,t)\right\|_{L^\infty(\Lambda_\varepsilon)}^2\right] \le \sum_{\omega=\frac{\epsilon^{-1}}{3}}^{\epsilon^{-1}}\frac{\epsilon^{-3}\gamma^2}{|\omega|^4} \simeq \gamma^2$$

For a given $R > 0$, Lemma 5.7 of [HW13] guarantees that, if $0 < \kappa < 1/4$ and

$\sup_{0\le s\le T}\|\mathsf{Ext}Z_\gamma(\cdot,s)\|_{L^\infty}\le R$, the following inequality holds

$$\left\|\mathsf{Ext}\bar{V}_\gamma(t)-\overline{\overline{V}}_\gamma(t)\right\|_{L^\infty([0,T]\times\mathbb{T})}\le C\big(\lambda,T,R,\left\|\mathsf{Ext}X^0_\gamma\right\|_{L^\infty}\big)\left(\gamma^\lambda+\left\|Z^{\text{high}}_\gamma\right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)}\right)$$

By Theorem 4.4.1 we have that

$$\limsup_{\gamma\to0}\mathbb{P}\Big(\sup_{0\le s\le T}\|\mathsf{Ext}Z_\gamma(\cdot,s)\|_{L^\infty}>R\Big)\to0\qquad\text{as } R\to\infty\,.$$

Sending $\gamma\to0$ and then $R\to\infty$ implies that, for all $l>0$

$$\begin{aligned}
&\limsup_{\gamma\to0}\mathbb{P}\left(\left|\sup_{0\le t\le T}\left\|\mathsf{Ext}\bar{V}_\gamma(t)-\overline{\overline{V}}_\gamma(t)\right\|_{L^\infty(\mathbb{T})}\right|>l\right)\\
&\quad\le\limsup_{R\to\infty,\gamma\to0}\ l^{-1}C\big(\kappa,\lambda,T,R,\left\|\mathsf{Ext}X^0_\gamma\right\|_{L^\infty}\big)\left(\gamma^\lambda+\mathbb{E}\left\|Z^{\text{high}}_\gamma\right\|_{L^\infty([0,T]\times\Lambda_\varepsilon)}\right)\\
&\quad+\limsup_{\gamma\to0}\mathbb{P}\Big(\sup_{0\le s\le T}\|\mathsf{Ext}Z_\gamma(\cdot,s)\|_{L^\infty}>R\Big)\\
&\quad=0
\end{aligned}$$

and this completes the proof. □

As a corollary of the previous propositions we obtain

Corollary 4.5.5 *For all $\lambda>0$ and $b>0$ we have*

$$\limsup_{\gamma\to0}\mathbb{P}\left(\sup_{0\le t\le T}\|V_\gamma(t)\|_{L^2(\Lambda_\varepsilon)}>b\gamma^{-\lambda}\right)=0$$

Proof. We decompose

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T}\|V_\gamma(t)\|_{L^2(\Lambda_\varepsilon)}>b\gamma^{-\lambda}\right)\\
&\le\mathbb{P}\left(\sup_{0\le t\le T}\left\|V_\gamma(t)-\bar{V}_\gamma(t)\right\|_{L^2(\Lambda_\varepsilon)}>\frac{b}{2}\gamma^{-\lambda}\right)+\mathbb{P}\left(\sup_{0\le t\le T}\left\|\bar{V}_\gamma(t)\right\|_{L^2(\Lambda_\varepsilon)}>\frac{b}{2}\gamma^{-\lambda}\right)
\end{aligned}$$

and the result is now a consequence of Propositions 4.5.4 and 4.5.1

$$\mathbb{P}\left(\sup_{0\le t\le T}\left\|\bar{V}_\gamma(t)\right\|_{L^2(\Lambda_\varepsilon)}>\frac{b}{2}\gamma^{-\lambda}\right)\le2b^{-1}\gamma^\lambda\mathbb{E}\left[\sup_{0\le t\le T}\left\|\bar{V}_\gamma(t)\right\|_{L^2(\Lambda_\varepsilon)}\right]\le C\gamma^\lambda$$

where the last expectation is finite because of Theorem 4.4.1 and Assumption 4.2.5. □

We would like to remark that the proof of Corollary 4.5.5 requires Theorem 4.4.1 but not Theorem 4.4.3.

4.6 Boltzmann-Gibbs principle

In this section we will present some heuristic discussions about the replacement lemmas we conjectured in Section 4.5. Our analysis is similar in spirit to another replacement lemma, commonly referred to as the *second order Boltzmann-Gibbs principle*, proposed by Jara and Gonçalves in [GJ13a, GJ14] for a class of particle systems with invariant Bernoulli product measure (see also [GJS15] for a generalization) which is a more precise and qualitative version of the original Boltzmann-Gibbs principle proved by Brox and Rost [BR84] (see [DMPSW86] for a proof in the reversible case and [CLO01] for the generalization to the stationary nonreversible case).
The principle states intuitively that fluctuation of the value of any local function around its mean is proportional to the fluctuation of the local density of particles, under a space time average.
Heuristically the principle is based on the idea that nonconserved local quantities of the dynamic tend to vanish, when averaged in space and time. Indeed, using the coercivity of the jump rates, we can see that the only locally conserved quantity for the dynamic is the local number of particle (magnetization in the language of spin systems). Therefore, under the action of the dynamic, every local quantity will be approximated by a function of the local density.
In particular, the fluctuation of every local quantity will approximately proportional to the fluctuation of the local density.
To deal with the situations when this proportionality constant becomes small or vanishes, in [GJ13a] the authors introduced a *Second order Boltzmann-Gibbs* principle, when they pushed the equivalence up to a second order in the fluctuation field. The proof of the Boltzmann-Gibbs principle relies on a spectral gap estimate on the dynamic restricted to small blocks and on an equivalence of ensembles for canonical and grand canonical measures.
One of the main features of the Kac-interaction is that it defines a mesoscopic scale in between the microscopic scale (given by the lattice) and the macroscopic system (given by the Cahn-Hilliard equation).
The purpose of this subsection is to prove a Boltzmann-Gibbs principle for non-product measures with smooth long range potential of the form (4.2), to replace functions varying over the microscopic scale with combination of the mesoscopic fluctuation field h_γ. Since most of the results hold in any dimension, in this section we shall assume $d \in \{1, 2\}$ and consider $\Lambda_N = \{-N+1, N\}^d$ and $\epsilon = N^{-1}$.

In order to better explain the nature of this replacement, we will now recall briefly the result in [GJ13a], which can be recast in terms of the Kawasaki dynamic presented in Section 4.1 when the inverse temperature $\beta = 0$. We will denote by π_m the Bernoulli product measure π_m on $\{-1, 1\}^{\Lambda_N}$ with mean m. For the choice $\beta = 0$, the interaction kernel κ_γ doesn't play any role for the dynamic and it is easy to see that the collection π_m is a family of invariant measures for the Kawasaki process at infinite temperature (β = 0) parametrized by their magnetization $m \in [-1, 1]$.
For $l \in \mathbb{N}$ define the cube

$$B_x^l = \{j \in \Lambda_N : |j - x|_\infty \leq l\}$$

and the magnetization inside the cube B_x^l as $\bar{\sigma}_x^l = \mathrm{Av}_{i \in B_x^l}\, \sigma_i$.
We recall that in the language of particle system, a function $f : \Sigma_N \to \mathbb{R}$ is said to be *local* if it depends on the value of the configuration in a finite number of sites, or equivalently that there exists $r > 0$ such that f depends only on the sites in B_0^r. This allows us to identify the function f even if the domain Σ_N is growing as $N \to \infty$.
Finally, for any local function f, we will use the notation

$$\Phi_f(m) \stackrel{\text{def}}{=} \mathbb{E}_{\pi_m}[f]\ .$$

We are now going to present the statement of the Boltzmann-Gibbs principle in case of the Kawasaki dynamic at $\beta = 0$.

Remark 4.6.1 In case $\beta = 0$, the density still evolves diffusively and therefore the correct rescaling of the time would be given by ϵ^{-2}. This is a crucial difference from the situation where β is converging to its critical value, where the scaling of the time is given by (4.20). This last case is actually an advantage from the point of view of convergence to equilibrium, since it implies that the local equilibrium of the density is reached much faster with respect to our time scale.

Let $f : \{-1, 1\}^{\Lambda_N} \to \mathbb{R}$ be a local function, recall the definition of $\Phi_f(m)$ above and let $G : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ a smooth test function.
In [DMPSW86, Theorem 1] it is shown, for a class of conservative dynamics satisfying a suitable mixing condition, that the fluctuation field

$$\epsilon^{\frac{d}{2}} \sum_{x \in \Lambda_N} \left\{\tau_x f(\sigma(\epsilon^{-2} s)) - \Phi_f(m)\right\} G(s, \epsilon x)$$

is approximated at first order by a linear function of the magnetization

$$\Phi'_f(m) \sum_{x\in\Lambda_N} \epsilon^{\frac{d}{2}} \big(\sigma_x(\epsilon^{-2}s) - m\big)\, G(s,\epsilon x)\ .$$

In the statement of the next theorems we abuse the notation writing

$$\begin{aligned}\tau_x f(\sigma(\epsilon^{-2}s)) - \Phi_f(m) - \Phi'_f(m)\big(\sigma_x(\epsilon^{-2}s) - m\big)& \\ = \big\{\tau_x f(\sigma) - \Phi_f(m) - \Phi'_f(m)\,(\sigma_x - m)\big\}\,(\epsilon^{-2}s)&\end{aligned}$$

Theorem 4.6.2 (Boltzmann-Gibbs principle) *For $\beta = 0$, let $f : \{0,1\}^{\Lambda_N} \to \mathbb{R}$ be any local function, and $G : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ a smooth function. Then we have that*

$$\mathbb{E}_{\pi_m}\left[\left(\int_0^T \epsilon^{\frac{d}{2}} \sum_{x\in\Lambda_N} \big\{\tau_x f(\sigma) - \Phi_f(m) - \Phi'_f(m)\,(\sigma_x - m)\big\}\,(\epsilon^{-2}s) G(s,\epsilon x)\, ds\right)^2\right]$$

vanishes in the limit $N \to \infty$.

In case $\Phi'_f(m) = 0$ and $d \geq 3$, a quantitative version of 4.6.2 when $\beta = 0$ is provided by [CLO01, Theorem 4.2] and can be stated using our notations as

Theorem 4.6.3 (Quantitative Boltzmann-Gibbs principle) *For $\beta = 0$, let $f : \{0,1\}^{\Lambda_N} \to \mathbb{R}$ be any local function satisfying $\Phi_f(m) = \Phi'_f(m) = 0$, and $G : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ a smooth function. Then there exists a constant $C(f,m,d)$ such that*

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{E}_{\pi_m}\left[\sup_{0\le t\le T}\left(\int_0^t \epsilon^{-1} \sum_{x\in\Lambda_N} \epsilon^{\frac{d}{2}} \tau_x f(\sigma(\epsilon^{-2}s))\, G(s,\epsilon x)\, ds\right)^2\right]& \\ \le C(f,m,d) \int_0^T \sum_{x\in\Lambda_N} \epsilon^d G(s,\epsilon x)\, ds\ .&\end{aligned}$$

The above result has been made more precise in [GJ14], where the authors were able to perform a local expansion up to the second order in the local density. The following result is proven for the Kawasaki dynamic at infinite temperature ($\beta = 0$) in $d = 1$, the generalization to any dimension being straightforward.

Theorem 4.6.4 (Second order Boltzmann-Gibbs principle) *For $\beta = 0$, let $f : \{0,1\}^{\Lambda_N} \to \mathbb{R}$ be a local function satisfying $\Phi_f(m) = \Phi'_f(m) = 0$, then there exists a constant*

$C(f,m,d)$ such that, for all $l \le \lfloor N/2 \rfloor$, $t \ge 0$ and a measurable function $G : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ we have

$$\mathbb{E}_{\pi_m}\left[\left(\int_0^t \sum_{x\in\Lambda_N} \epsilon^{\frac{d}{2}-1}\tau_x\left\{f(\sigma) - \frac{\Phi_f''(m)}{2}\mathcal{Q}_m(l,\sigma)\right\}(\epsilon^{-2}s)G(s,\epsilon x)\,ds\right)^2\right]$$
$$\le C(f,m,d)\left(\epsilon^2\,\mathbf{r}(l) + \frac{T}{l^{2d}}\right)\int_0^T \sum_{x\in\Lambda_N}\epsilon^d\,G^2(s,\epsilon x)\,ds \quad (4.75)$$

where $\mathcal{Q}_m(l,\sigma(s))$ is the quadratic field

$$\mathcal{Q}_m(l,\sigma) = \left(\bar{\sigma}_0^l - m\right)^2 - \frac{1-m^2}{(2l+1)^d}$$

and

$$\mathbf{r}(l) = \begin{cases} l & \text{if } d = 1 \\ \log(l) & \text{if } d = 2 \\ 1 & \text{if } d \ge 3 \end{cases}$$

The second order Boltzmann-Gibbs principle allows to replace any local function f with the quadratic function of the local density $\mathcal{Q}_m(l,\sigma)$ "weakly locally" (in the sense of [KL99, Def 3.0.2]. The radius l of the block in (4.75) can then be chosen proportional to ϵ^{-1} for $\mathcal{Q}_m(l,\sigma)$ to be a quadratic function of the density in a macroscopic interval around the origin. In this way it is possible to replace the effect of the local function f with a quadratic function of the macroscopic density.
Theorem 4.6.4 has been used by M.Jara and P.Gonçalves in [GJ14] to define the concept of *energy solution* for the KPZ equation (1.3), later perfected in [GJ13b].

The previous theorems have been proven for a restricted class of models so far (see [GJS15]). This is because the proofs require the dynamic to enjoy good ergodic proprieties on small domains, and the invariant measure to satisfy good mixing proprieties. The hypothesis over the invariant measures are given in terms of the specification for the Gibbs measure [DG74, Eq. 2.8] or decay of the correlations [VY97, Eq. 2.7], that guarantees that the DLR conditions for the specification of the Gibbs measure are satisfied. This usually forces to work with Gibbs measure having weak interaction (small β) and finite range $\gamma^{-1} \le C$ with C independent of N.

The Boltzmann-Gibbs principle presented in [GJ14, GJ13a] has the disadvantage that it requires the dynamic to start from the invariant measure because of the use of the

Kipnis-Varadhan inequality (Proposition 4.6.5). On the other hand it yields very precise bounds in L^2, and it allows the replacement to hold up to small macroscopic sizes $l \ll \epsilon^{-1}$. The aim of the rest of the chapter is to prove that Theorem 4.6.4 holds for the Kawasaki process with $\beta > 0$ introduced in Section 4.1, up to the point when the radius of the block l is mesoscopic.

The main purpose of the replacement lemma is to prove Proposition 4.5.4. It is important to remark that the arguments we are going to describe in the rest of the chapter do not constitute a proof for it.

We will need to apply the Boltzmann-Gibbs principle to argue that the fluctuation $\delta^{-2}\sigma_x(\alpha^{-1}s)\sigma_{x+1}(\alpha^{-1}s)$ are locally proportional to a quadratic form of the "mesoscopic" fluctuation of the magnetization the field X_γ of the form

$$(1+\mathfrak{g}_\gamma)X_\gamma(s,z)X_\gamma(s,z+\epsilon) - \delta^{-2}\mathfrak{g}_\gamma$$

where $\mathfrak{g}_\gamma$ has been defined in (4.22).

The replacement lemma that we are going to prove differs from Theorem 4.6.4 for the following reasons: using the Feynman-Kac formula instead of the Kipnis-Varadhan lemma, we are going to allow the process to start from any initial condition at the cost of obtaining convergence in probability and not in L^2. The key facts that allow to obtain the result are the large time scale and the fact that we are not pushing the size of the block to the macroscopic scale $\sim \epsilon^{-1}$, but only to mesoscopic scale $\sim \gamma^{-1}$. The approach that we are going to use has been mainly inspired by the proof of the second order Boltzmann-Gibbs principle in [GJ14], the proof the super exponential estimate in [KOV89] and the works of Quastel [Qua96] and Rezakhanlou [Rez94].

4.6.1 A separation of scales

The main feature of the measure that we are going to exploit is the fact that, when we look at the canonical measure conditioned on blocks with size much smaller than the interaction length γ^{-1}, the measure behaves like a product measure conditioned to have a fixed internal magnetization. This is the section where we make use the assumption (FLAT) over the interaction $\mathfrak{K}$. The length $\gamma^{-1}\mathbf{a} > 0$ in (FLAT) represents, heuristically speaking, the point where a small block ceases to be "microscopic" and start to be "mesoscopic". In principle one could drop this assumption, and use only the smoothness of $\mathfrak{K}$, many formulas however are not amenable to calculations, and we found this simplification convenient.

First recall some general theorem for general Markov processes with countable state space E. Let μ be an invariant measure for a Markov process X_t, and let L and L^s be, respectively, the generator of the process and its symmetric part. For a function g in $L^2(\mu)$, define the norm $\|g\|_{\mathcal{H}_L^{-1}(\mu)}$ with the variational formula

$$\|g\|^2_{\mathcal{H}_L^{-1}(\mu)} := \sup_{f \text{ local}} \{\mathbb{E}_\mu[gf] - \mathbb{E}_\mu[f(-L^s)f]\}$$

The next inequality is a fundamental tool in the general theory o Markov processes.

Proposition 4.6.5 (Kipnis-Varadhan) *Let X_t be a Markov process with countable state space E and let μ be a measure on the state space E invariant for the process. Then, there exists a constant $C > 0$ such that, for all $g : [0,T] \to L^2(E,\mu)$*

$$\mathbb{E}_\mu\left[\sup_{0\le t\le T}\left(\int_0^t g(s,X_s)\,ds\right)^2\right] \le C\int_0^T \|g(s,\cdot)\|^2_{\mathcal{H}_L^{-1}(\mu)}\;ds\;.$$

See [KL99, Appendix 1, Proposition 6.1] for a proof. In particular the theorem doesn't assume the process to be reversible with respect to μ.

The major advantage of the previous theorem is the fact that, for the case in which L^s is the generator of the simple exclusion process ($\beta = 0$), it is possible to prove that functions with disjoint support are orthogonal with respect to the $\|\cdot\|_{\mathcal{H}_L^{-1}(\mu)}$ norm. This is made precise in the following proposition, which is proven in [GJ14, Proposition 3.4].

Proposition 4.6.6 *Assume L is the generator of the Kawasaki dynamic with $\beta = 0$, π_m is the Bernoulli product measure and let g_i for $i \in I$, be a collection of local functions with disjoint support, then*

$$\left\|\sum\nolimits_{i\in I} g_i\right\|^2_{\mathcal{H}_L^{-1}(\pi_m)} \le \sum\nolimits_{i\in I}\|g_i\|^2_{\mathcal{H}_L^{-1}(\pi_m)}\;.$$

For convenience we are going to sketch the proof.

Proof. We recall the fact that L is symmetric and therefore $L^s = L$. Let $S_i \subseteq \Lambda_N$ the support of g_i, then

$$\mathbb{E}_{\pi_m}\sum_{i\in I}\sum_{\substack{a,b\in S_i\\|a-b|=1}}\big(f(\sigma^{a,b}) - f(\sigma)\big)^2 \le \mathbb{E}_{\pi_m}\big[f(-Lf)\big]$$

Let $g = \sum_{i\in I} g_i$. From the definition of $\|\cdot\|_{\mathcal{H}_L^{-1}(\mu)}$ we have

$$\Big\|\sum_{i\in I} g_i\Big\|^2_{\mathcal{H}_L^{-1}(\pi_m)} = \sup_{f \text{ local}} \{\mathbb{E}_{\pi_m}[gf] - \mathbb{E}_{\pi_m}[f(-L)f]\}$$
$$\le \sup_{f \text{ local}} \sum_{i\in I} \Big\{\mathbb{E}_{\pi_m}[g_i f] - \mathbb{E}_\mu\Big[\sum_{\substack{a,b\in S_i\\ |a-b|=1}} \big(f(\sigma^{a,b}) - f(\sigma)\big)^2\Big]\Big\} .$$

Let $h_i \stackrel{\text{def}}{=} \mathbb{E}_{\pi_m}[f|\mathcal{F}_{S_i}]$ be the projection of f on the σ-algebra generated by the spins in S_i, since $g_i \in \mathcal{F}_{S_i}$ we have that $\mathbb{E}_{\pi_m}[g_i f] = \mathbb{E}_{\pi_m}[g_i h_i]$.

$$\mathbb{E}_{\pi_m}\Big[\sum_{\substack{a,b\in S_i\\ |a-b|=1}} \Big(f(\sigma^{\{a,b\}}) - f(\sigma)\Big)^2\Big]$$
$$\ge \mathbb{E}_{\pi_m}\Big[\sum_{\substack{a,b\in S_i\\ |a-b|=1}} \Big(h_i(\sigma^{\{a,b\}}) - h_i(\sigma)\Big)^2\Big] = \mathbb{E}_{\pi_m}[h_i(-Lh_i)] .$$

Using the above estimates we obtain

$$\Big\|\sum_{i\in I} g_i\Big\|^2_{\mathcal{H}_L^{-1}(\pi_m)} \le \sup_{f \text{ local}} \sum_{i\in I} \Big\{\mathbb{E}_{\pi_m}[g_i f] - \mathbb{E}_\mu\Big[\sum_{\substack{a,b\in S_i\\ |a-b|=1}} \big(f(\sigma^{a,b}) - f(\sigma)\big)^2\Big]\Big\}$$
$$\le \sum_{i\in I} \sup_{\substack{h_i = \mathbb{E}_{\pi_m}[f|\mathcal{F}_{S_i}]\\ f \text{ local}}} \Big\{\mathbb{E}_{\pi_m}[g_i h_i] - \mathbb{E}_{\pi_m}[h_i(-Lh_i)]\Big\}$$
$$\le \sum_{i\in I} \sup_{h \text{ local}} \Big\{\mathbb{E}_{\pi_m}[g_i h] - \mathbb{E}_{\pi_m}[h(-Lh)]\Big\}$$
$$\le \sum_{i\in I} \|g_i\|^2_{\mathcal{H}_L^{-1}(\pi_m)} .$$

□

This is a key lemma in the proof of the second order Boltzmann-Gibbs principle, and the aim is to extend it to the case of the Kawasaki dynamic for any $\beta > 0$ and Kac potential. The result we where able to prove is however restricted to the case of local functions having support of diameter much smaller than the interaction length of the Kac potential, but this turns out to be sufficient for our purposes.

Remark 4.6.7 Let $l \in \mathbb{N}$ with $l < \mathbf{a}/2$ and denote with $\Lambda \subseteq \Lambda_N$ a subset of radius $\le l$. Let $\eta \in \Sigma_N$ and for $M \in \{-1, -1 + \frac{2}{|\Lambda|}, \ldots, 1 - \frac{2}{|\Lambda|}, 1\}$ denote the Canonical Gibbs measure on Λ constrained to have magnetization M and external field η

$$\mu_\gamma^{\Lambda,\eta,M}(g) = \mu_\gamma^{\Lambda,\eta}\Big[g\Big|\operatorname{Av}_{i\in\Lambda}\sigma_i = M\Big] .$$

From the form of the measure and (FLAT) we have that, for $\sigma \in \{-1,+1\}^\Lambda$ and $\mathrm{Av}_{i\in\Lambda}\,\sigma_i = M$

$$\mathcal{H}_\gamma^{\Lambda,\eta}(\sigma) = \frac{\beta}{2}\sum_{x,y\in\Lambda}\kappa_\gamma(x,y)\sigma_x\sigma_y + \beta\sum_{x\in\Lambda}\sigma_x\alpha_\gamma(x,\eta)$$
$$= \frac{\beta}{2}\kappa_\gamma(1)M\left(M-\frac{1}{|\Lambda|}\right)|\Lambda|^2 + \beta\sum_{x\in\Lambda}\sigma_x\alpha_\gamma(x,\eta)$$

and $\mu_\gamma^{\Lambda,\eta,M}$ coincides with the conditioned inhomogeneous product measure over Λ with tilting

$$\beta\alpha_\gamma(x,\eta) = \beta\sum_{z\in\Lambda_N\setminus\Lambda}\kappa_\gamma(x,z)\eta_z$$

which is uniformly bounded in absolute value.

The next proposition takes advantage of the particular form of the Gibbs measure and is a consequence of Remark 4.6.7.

Proposition 4.6.8 *Let $\Lambda \subseteq \Lambda_N$ with $\mathrm{diam}(\Lambda) \le \mathbf{a}\gamma^{-1}$ and M be as in Remark 4.6.7 and consider $\mu_\gamma^{\Lambda,\eta,M}$ be the canonical Gibbs measure on Λ with boundary condition η and conditioned to have $\mathrm{Av}_{i\in\Lambda}\,\sigma_i = M$. Let $\pi^{\Lambda,M}$ be the homogeneous Bernoulli product measure over the configurations in $\{-1,1\}^\Lambda$ conditioned to satisfy $\mathrm{Av}_{i\in\Lambda}\,\sigma_i = M$.*
Then there exists C, independent of l and γ, such that

$$\frac{1}{C^{l^d\gamma}} \le \frac{d\mu_\gamma^{\Lambda,\eta,M}}{d\pi^{\Lambda,M}}(\sigma) \le C^{l^d\gamma} \qquad \pi^{\Lambda,M} - a.s.$$

where $\frac{d\mu_\gamma^{\Lambda,\eta,M}}{d\pi^{\Lambda,M}}$ is the Radon-Nikodym derivative.
If moreover $l^d \le c_0\gamma^{-1}$, it follows that there exists $C' = C'(c_0)$ such that, for every $f : \{-1,1\}^\Lambda \to \mathbb{R}$

$$\mu_\gamma^{\Lambda,\eta,M}\left(f-\mu_\gamma^{\Lambda,\eta,M}[f]\right)^2 \le C'\pi^{\Lambda,M}\left(f-\pi^{\Lambda,M}[f]\right)^2 . \tag{4.76}$$

We would like to stress that (4.76) is expected to hold for any $l \le c_0\gamma^{-1}$, but since we are interested in the one dimensional case, this result is sufficient.

Proof. By Remark 4.6.7, we have that $\mu_\gamma^{\Lambda,\eta,M}$ is an inhomogeneous product measure conditioned to have internal magnetization M, therefore, for any $\lambda \in \mathbb{R}$ we have

$$\left|\log\frac{d\mu_\gamma^{\Lambda,\eta,M}}{d\pi^{\Lambda,M}}(\sigma)\right| \le 2\beta\sum_{x\in\Lambda}\left|\sum_{z\in\Lambda_N\setminus\Lambda}\kappa_\gamma(x,z)\eta_z - \lambda\right|$$

from the smoothness of $\mathfrak{K}$ we have for any $x, y \in \Lambda_N$

$$\left| \sum_{z \in \Lambda_N \setminus \Lambda} \kappa_\gamma(x,z) - \kappa_\gamma(y,z)\eta_z \right| \lesssim \gamma |x-y|$$

and the proposition follows using the fact that $\mathrm{diam}(\Lambda) \leq l$. □

4.6.2 Spectral gap for the Kawasaki dynamic in small blocks

The next result is a classic result in the context of the Kawasaki dynamic restricted in a finite box (see [LY93]). We need to remark however that our particular Ising-Kac model doesn't seem to be covered by the classic literature. In [LY93] for instance, the Hamiltonian is assumed to have finite range of interaction. While this condition is satisfied for fixed γ, we need the constant of the spectral gap inequality to stay bounded as the range of the Hamiltonian goes to infinity.

Given $\Lambda \subseteq \Lambda_N$, and two configurations $\sigma \in \{-1,1\}_\Lambda$ and $\eta \in \{-1,1\}^{\Lambda_N}$, let us define

$$(\sigma \sqcup_\Lambda \eta)_x = \begin{cases} \sigma_x & \text{for } x \in \Lambda \\ \eta_x & \text{for } x \notin \Lambda \end{cases}$$

Let $\mathscr{L}^K_{\Lambda,\eta}$ be the generator of the Kawasaki dynamic (4.4) constrained to have only exchanges between sites in Λ and where the configuration η is used for the sites in (4.3) outside of Λ

$$\mathscr{L}^K_{\Lambda,\eta} g(\sigma) = \frac{1}{2} \sum_{\substack{(x,y)\in\Lambda\times\Lambda \\ |x-y|=1}} c^K_\gamma(x,y,\sigma \sqcup_\Lambda \eta)\left(g(\sigma^{x,y}) - g(\sigma)\right) \qquad \sigma \in \{-1,1\}^\Lambda .$$

An important observation is that, if $\mathrm{diam}(\Lambda) \leq \mathbf{a}\gamma^{-1}$, the difference $\tilde{h}_\gamma(y) - \tilde{h}_\gamma(x)$ in (4.3) only depends on η and not on the internal configuration $\sigma \in \{-1,1\}^\Lambda$. Since the rates satisfy the coercivity condition (CB), $\mu_\gamma^{\Lambda,\eta,M}$ are the only ergodic measure on $\{-1,+1\}^\Lambda$ with respect to $\mathscr{L}^K_{\Lambda,\eta}$.

Let $\mathbf{D}_\gamma^{\Lambda,\eta,M}$ be the Dirichlet form for the block Λ, external configuration η, with respect to the canonical measure, is defined as

$$\mathbf{D}_\gamma^{\Lambda,\eta,M}(f) = 2\mu_\gamma^{\Lambda,\eta,M}\left[f(-\mathscr{L}^K_{\Lambda,\eta} f)\right] . \tag{4.77}$$

We will denote with $\mathbf{D}_\gamma^{\Lambda,\eta}$ the same quantity as above, with the expectation taken with respect to the Grand Canonical measure $\mu_\gamma^{\Lambda,\eta}$ and we will denote with $\mathbf{D}_\gamma^{\Lambda_N}$ the Dirichlet form in Λ_N taken with respect to the full generator (4.4).

The right-hand-side of (4.77) is a quadratic form in $L^2\left(\mu_\gamma^{\Lambda,\eta,M}\right)$ and can be used to define a norm over the subspace space of $L^2\left(\mu_\gamma^{\Lambda,\eta,M}\right)$ orthogonal to the constant functions.
It will be also useful to consider the dual of the norm in (4.77) with respect to the $L^2\left(\mu_\gamma^{\Lambda,\eta,M}\right)$ inner product. For a function $f \in L^2\left(\mu_\gamma^{\Lambda,\eta,M}\right)$ with $\mu_\gamma^{\Lambda,\eta,M}[f] = 0$, let

$$\mathbf{V}_\gamma^{\Lambda,\eta,M}(f) = \sup_{g \text{ local}} \left\{ 2\, \mu_\gamma^{\Lambda,\eta,M}[f\, g] - \mu_\gamma^{\Lambda,\eta,M}\left[g(-\mathscr{L}_{\Lambda,\eta}^K g)\right] \right\} . \tag{4.78}$$

One of the tools needed for the Boltzmann-Gibbs principle is an estimate on the spectral gap of the operator $\mathscr{L}_{\Lambda,\eta}^K$. If the radius of the box is small enough, by Remark 4.6.7, the spectral gap inequality follows from the result in [Qua96], proven for inhomogeneous product measure. The next result is valid in any dimension, but we are only going to prove it in the case of dimension 1.

Proposition 4.6.9 *Let $\Lambda \subseteq \Lambda_N$ be a cube of radius $0 < l < \mathbf{a}\gamma^{-1}$. Let f be a local function with support in Λ. Then, for all $\beta \in \mathbb{R}$, there exists a constant $C = C(\beta) > 0$, independent of $f, M, \eta, l, \Lambda, \gamma$, such that*

$$\mu_\gamma^{\Lambda,\eta,M}\left[\left(f - \mu_\gamma^{\Lambda,\eta,M}[f]\right)^2\right] \le C l^{-2} \mathbf{D}_\gamma^{\Lambda,\eta,M}(f) \tag{4.79}$$

As a consequence of the previous result we have that

$$\mathbf{V}_\gamma^{\Lambda,\eta,M}(f) \le C l^2 \mu_\gamma^{\Lambda,\eta,M}\left[\left(f - \mu_\gamma^{\Lambda,\eta,M}[f]\right)^2\right] \tag{4.80}$$

In virtue of [LY93], the result is expected to hold for arbitrarily big blocks, however the restriction to blocks of order $\mathbf{a}\gamma^{-1}$ is sufficient for our purposes and easier to prove. The following is referred as the moving particle lemma in [Qua96]

Lemma 4.6.10 (Moving particle lemma) *Let c_0 a positive real number and let $\beta \in \mathbb{R}$. Then there exists a constant $C = C(\beta, c_0) > 0$ such that for any $\Lambda \subseteq \Lambda_N$ cube of radius $0 < l < c_0\gamma^{-1}$, any f local function with support in Λ and for all $x < y \in \Lambda$, the following inequality*

$$\mu_\gamma^{\Lambda,\eta,M}\left[\left(f(\sigma^{x,y}) - f(\sigma)\right)^2\right] \le C|x-y| \sum_{j=x}^{y-1} \mu_\gamma^{\Lambda,\eta,M}\left[\left(f(\sigma^{j,j+1}) - f(\sigma)\right)^2\right] \tag{4.81}$$

holds. We remark that the constant C is independent of $f, M, \eta, l, \Lambda, \gamma$.

Proof. In the following proof, C will denote a generic constant which might be different from line to line. Define $\tau_{x,y}$ as $\tau_{x,y} f(\sigma) = f(\sigma^{x,y})$. In particular it is possible to write the operator $\tau_{x,y}$ as the composition of $\tau_{j,j+1}$ for $j \in \{x, x+1, \dots, y-2, y-1\}$. Assume, for notational simplicity, that $x = 1$ and $y = k$.

$$\tau_{1,k} = \tau_{1,2} \cdots \tau_{k-2,k-1}\ \tau_{k-1,k} \cdots \tau_{2,3}\tau_{1,2}\ .$$

Let us define, for brevity, $\tau^{(j)}$ as the j-th operator appearing above, in such a way that $\tau_{1,k} = \prod_{i=1}^{2k-3} \tau^{(i)}$. Then

$$f(\sigma^{1,k}) - f(\sigma) = \sum_{j=1}^{2k-4} \left(\prod_{i=j}^{2k-3} \tau^{(i)} f(\sigma) - \prod_{i=j+1}^{2k-3} \tau^{(i)} f(\sigma) \right)$$

and

$$\mu_\gamma^{\Lambda,\eta,M}\left[\left(f(\sigma^{1,k}) - f(\sigma) \right)^2 \right] \\ \leq (2k-4) \sum_{j=1}^{2k-4} \mu_\gamma^{\Lambda,\eta,M}\left[\left(\prod_{i=j}^{2k-3} \tau^{(i)} f(\sigma) - \prod_{i=j+1}^{2k-3} \tau^{(i)} f(\sigma) \right)^2 \right]$$

the last expectations is given by

$$\sum_{\sigma\in\Sigma_\Lambda} \mu_\gamma^{\Lambda,\eta,M}(\sigma) \left(\tau^{(j)} f\left(\prod_{i=j+1}^{2k-3} \tau^{(i)} \sigma \right) - f\left(\prod_{i=j+1}^{2k-3} \tau^{(i)} \sigma \right) \right)^2 \\ = \sum_{\sigma\in\Sigma_\Lambda} \mu_\gamma^{\Lambda,\eta,M}\left(\prod_{i=2k-3}^{j+1} \tau^{(i)} \sigma \right) \left(\tau^{(j)} f(\sigma) - f(\sigma) \right)^2$$

where the last line is obtained changing $\sigma \mapsto \prod_{i=2k-3}^{j+1} \tau^{(i)} \sigma$. Now the proof is complete using iteratively the bound

$$\mu_\gamma^{\Lambda,\eta,M}(\tau_{j,j+1}) \leq e^{C|\beta|\gamma} \mu_\gamma^{\Lambda,\eta,M}(\sigma)$$

which follows from the form of the Gibbs measure (4.2) and the kernel (4.9)

$$|\mathcal{H}_\gamma^{\Lambda,\eta}(\sigma^{x,y}) - \mathcal{H}_\gamma^{\Lambda,\eta}(\sigma)| \leq 2|\beta||h_\gamma(x) - h_\gamma(y)| \leq C|\beta|\gamma|x-y|\ .$$

□

4.6.3 Equilibrium fluctuation

Recall the definition of μ_γ, in (4.2) which is an invariant and reversible measure for the Kawasaki dynamic on the periodic lattice and, for any $\sigma \in \{-1,1\}^{\Lambda_N}$, recall also the definition of $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$ the Canonical Gibbs measure in the block B_x^l, conditioned to have magnetization $\bar{\sigma}_x^l$ and boundary condition σ. Recall moreover that $\sigma(s)$ represents the spin configuration at time $s \in \mathbb{R}^+$ under the Kawasaki dynamic. In this section we are going to provide an argument towards the proof of Proposition 4.5.4 under the (very restrictive) assumption that the Kawasaki process is starting at the reversible measure μ_γ and recall the scaling (4.21).

We will do so for two reasons: firstly the arguments in the equilibrium case is simpler and can provide some guidelines towards the general case, secondly in the equilibrium case the results are more robust and the same procedure might also be applicable to the case of the Kawasaki dynamic in the two dimensional torus. For this reason in this subsection we will keep the dependence on the dimension d explicit and make use of the scaling (4.21).

As it is classic in the theory of particle systems, the replacement lemma consists of two main steps (see Chapter 5, Section 3 of [KL99] for an example). In the first step, the function $\sigma_x\sigma_{x+1}$ is replaced with an average of the spins in a microscopic block around x and this is usually referred as "one block estimate". In the second step the average of the spin in a large microscopic block is compared with the average of the spins inside a small macroscopic block. This second step is usually referred as "two blocks estimate".

We will first introduce a preliminary technical proposition. Let $\mathbf{a}$ the constant defined in (FLAT). The next proposition is essentially a version of [GJ14, Cor. 3.5] which is tailor-made for our problem.

Proposition 4.6.11 *Let $0 < l \leq \mathbf{a}\gamma^{-1}$, and let $f_{x,l} : \Sigma_N \to \mathbb{R}$ for $x \in \Lambda_\varepsilon$ to be a family of functions, having zero mean with respect to any Canonical Gibbs measure in the block B_x^l*

$$\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[f_{x,l}] = 0$$

and satisfying $|f_{x,l}(\sigma)| \leq 1$. Let $G : \mathbb{R}_+ \times \mathbb{T}^d \to L^2(\mu_\gamma)$ be such that for all $(s,x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $G(s,x)$ is measurable with respect to the σ-algebra generated by $\{\sigma_i : i \notin B_x^l\}$.

Then there exists a constant $C > 0$ such that

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{0\le t\le T}\left(\int_0^t \epsilon^d \sum_{x\in\Lambda_N} f_{x,l}(\sigma(\alpha^{-1}s))\; G(s,\epsilon x)(\sigma(\alpha^{-1}s))\; ds\right)^2\right] \tag{4.82}$$
$$\le C\alpha\epsilon^d l^d \mathbb{E}_{\mu_\gamma}\left[\int_0^T \epsilon^d \sum_{x\in\Lambda_N} \mathbf{V}_\gamma^{B_x^l,\sigma(\alpha^{-1}s),\bar\sigma_x^l(\alpha^{-1}s)}\left[f_{x,l}\right]\; G^2(s,\epsilon x)(\sigma(\alpha^{-1}s))ds\right]$$

Moreover, if $f_{x,l}$ satisfies

$$\mathrm{Var}_\beta^{B_x^l,\eta,M}\left[f_{x,l}\right] \le C_1 l^{-\vartheta}$$

for a constant C_1 independent of η, M, l, x, we have that the quantity above is bounded by

$$CC_1\alpha\epsilon^d l^{d+2-\vartheta}\mathbb{E}_{\mu_\gamma}\left[\int_0^T \epsilon^d \sum_{x\in\Lambda_N} G^2(s,\epsilon x)(\sigma(\alpha^{-1}s))\; ds\right] \tag{4.83}$$

Proof. Using the assumption on $f_{x,l}$ and G, we have that the expectation

$$\mu_\gamma\left[f_{x,l}(\sigma)\; G(s,\epsilon x)(\sigma)\right] = \mu_\gamma\left[\mu_\gamma^{B_x^l,\sigma,\bar\sigma_x^l}\left[f_{x,l}(\sigma)\right]\; G(s,\epsilon x)(\sigma)\right] = 0$$

we can then apply the Kipnis-Varadhan lemma given in Proposition 4.6.5 with the generator of the Kawasaki process speeded up by a factor α^{-1} to the left-hand-side of (4.82), to obtain the bound

$$\int_0^T \sup_{g\text{ local}} \mu_\gamma\left[2g(\sigma)\;\epsilon^d \sum_{x\in\Lambda_N} f_{x,l}(\sigma)\; G(s,\epsilon x)(\sigma) - \alpha^{-1}g(\sigma)(-\mathscr{L}^K g)(\sigma)\right]\; ds\,.$$

It is immediate to see, counting the bonds on the lattice Λ_N, that

$$\mu_\gamma[g(-\mathscr{L}^K)g] \ge (2l+1)^{-d}\sum_{x\in\Lambda_N}\mu_\gamma[g(-\mathscr{L}^K_{B_x^l,\sigma})g]$$

and therefore the calculation

$$\mu_\gamma\left[2g(\sigma)\;\epsilon^d \sum_{x\in\Lambda_N} f_{x,l}(\sigma)\; G(s,\epsilon x)(\sigma) - \alpha^{-1}g(\sigma)(-\mathscr{L}^K g)(\sigma)\right]$$
$$\le \mu_\gamma\left[2\epsilon^d \sum_{x\in\Lambda_N} G(s,\epsilon x)(\sigma)\mu_\gamma^{B_x^l,\sigma,\bar\sigma_x^l}\left[f_{x,l}\; g\right] - \frac{\mu_\gamma^{B_x^l,\sigma,\bar\sigma_x^l}\left[g(-\mathscr{L}^K_{B_x^l,\sigma}g)\right]}{\alpha(2l+1)^d}\right]$$

in the last inequality we used the fact that $G(s,\epsilon x)(\sigma)$ doesn't depend on the spins in B_x^l and

it is $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$-a.s constant. Since $\mu_\gamma^{B_x^l,\eta,M}[f_{x,l}] = 0$ we can use the definition in (4.78) to obtain (4.82). In particular, proof uses only the fact that $G(s,\epsilon x)(\sigma)$ is $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$-a.s constant, so the conclusion holds also in case $G(s,\epsilon x)(\sigma)$ is a function of the total magnetization inside the block B_x^l.
In order to obtain (4.83) it is sufficient to apply (4.80) and the bound over the variance in the assumption. □

We are now ready to introduce the main propositions of this Subsection, which are clearly inspired by Lemmas 4.3, 4.4 and 4.5 in [GJ14].

Consider a test function $\phi : \Lambda_\varepsilon \times \mathbb{R}_+ \to \mathbb{R}$: we will often think of ϕ as being

$$\phi(x,s) = P_{t-s}^{K,\gamma} K_\gamma(z-x) 1_{\{s<t\}}$$

and or a discretization of a continuous function defined on the torus $\mathbb{T}$.
For a configuration $\sigma \in \Sigma_N$, we define for the following propositions the quantity

$$\Psi_x^l(\sigma) \stackrel{\text{def}}{=} \mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[\sigma_x \sigma_{x+1}] \tag{4.84}$$

The first step of the procedure is the following proposition that, in the spirit of [GJ14] will be called one block estimate. As remarked before we will work in dimension $d = 1$ but we will keep the dependence of the dimension explicit, because the same proof holds in dimension $d = 2$.

Proposition 4.6.12 (One block estimate) *For any $l_0 > 0$ and sufficiently small γ, there exists $C > 0$ such that*

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{0\le t\le T}\left(\int_0^t \epsilon^d \sum_{x\in\Lambda_N} \nabla_\epsilon\, \phi(\epsilon x, s)\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+ \tilde{h}(x,\alpha^{-1}s)\right)\right.\right.$$
$$\left.\left.\times\, \delta^{-2}\left(\sigma_x(\alpha^{-1}s)\sigma_{x+1}(\alpha^{-1}s) - \Psi_x^{l_0}(\sigma(\alpha^{-1}s))\right)\, ds\right)^2\right]$$
$$\le C\alpha\epsilon^d l_0^{d+2}\delta^{-4}\int_0^T \mathbb{E}_{\mu_\gamma}\left\|\nabla_\epsilon\, \phi(\cdot,s)\, \nabla_\epsilon\, X_\gamma(\cdot,s)\right\|_{L^2(\Lambda_\varepsilon)}^2\, ds$$

Proof. The proof is an application of Proposition 4.6.11 for

$$f_{x,l_0}(\sigma) = \delta^{-2}\left(\sigma_x\sigma_{x+1} - \Psi_x^{l_0}(\sigma)\right) ,$$

$$G(s,\epsilon x)(\sigma) = \delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}(x,s)\right)$$

$$\mathrm{Var}_\beta^{B_x^{l_0},\eta,M}\left[f_{x,l}\right] \le 2\delta^{-4}$$

in particular, by definition, $\mu_\gamma^{B_x^{l_0},\sigma,\bar{\sigma}_x^{l_0}}[f_{\epsilon x}(\sigma)] = 0$ and by (FLAT) we have that $\nabla_N^+\tilde{h}(0)$ doesn't depend on the spins in $B_0^{l_0}$, if $l_0 \le \mathbf{a}\gamma^{-1}$

$$\nabla_N^+\tilde{h}(0) = \sum_{i\in\Lambda_N\setminus B_0^{l_0}} [\tilde{\kappa}_\gamma(1-i) - \tilde{\kappa}_\gamma(-i)]\sigma_i \,.$$

□

In a similar way one can show the following

Proposition 4.6.13 (Renormalization step) *For $l \le \mathbf{a}\gamma^{-1/d}$, there exists $C > 0$ such that*

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{0\le t\le T}\left(\int_0^t \epsilon^d \sum_{x\in\Lambda_N} \nabla_\epsilon\,\phi(\epsilon x, s)\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,\alpha^{-1}s)\right)\right.\right.$$
$$\left.\left.\times\,\delta^{-2}\left(\Psi_x^l(\sigma,\alpha^{-1}s) - \Psi_x^{2l}(\sigma,\alpha^{-1}s)\right)\,ds\right)^2\right]$$
$$\le C\alpha\epsilon^d l^{-d+2}\delta^{-4}\mathbb{E}_{\mu_\gamma}\int_0^T \|\nabla_\epsilon\,\phi(s)\,\nabla_\epsilon\,X_\gamma(s)\|^2_{L^2(\Lambda_\varepsilon)}\,ds$$

Proof. The proof is identical to the proof of Proposition 4.6.12 with the only difference that

$$f_{x,l}(\sigma) = \delta^{-2}\left(\Psi_x^l(\sigma) - \Psi_x^{2l}(\sigma)\right)$$

and for all $\eta \in \Sigma_N$ we use the fact that, by (4.76)

$$\mu_\gamma^{B_x^{2l},\eta}\left[\left(\Psi_x^l(\sigma) - \Psi_x^{2l}(\sigma)\right)^2\right] \le C_1 l^{-2d}$$

□

The next proposition completes Proposition 4.6.12 performing the replacement up to a block of radius proportional to γ^{-1} in dimension 1.

Proposition 4.6.14 (Two blocks estimate) *For $l \leq \mathbf{a}\gamma^{-1/d}$ there exists a $C >$ such that*

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{0\leq t\leq T}\left(\int_0^t \epsilon^d \sum_{x\in\Lambda_N} \nabla_\epsilon\,\phi(\epsilon x,s)\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,\alpha^{-1}s)\right)\right.\right.$$
$$\left.\left.\times\,\delta^{-2}\left(\Psi_x^{l_0}(\sigma,\alpha^{-1}s)-\Psi_x^{l}(\sigma,\alpha^{-1}s)\right)ds\right)^2\right]$$
$$\leq C\alpha\epsilon^d\delta^{-4}\mathbf{r}_d(l)\int_0^T \mathbb{E}_{\mu_\gamma}\left\|\nabla_\epsilon\,\phi\,\nabla_\epsilon\,X_\gamma\right\|^2_{L^2(\Lambda_\varepsilon)}(s)\,ds \quad (4.85)$$

where

$$\mathbf{r}_d(l)=\begin{cases} l & \text{if } d=1\\ \log^2(l) & \text{if } d=2\end{cases}$$

Proof. We will prove the proposition for l of the form $l=l_0 2^J$, the general case follows from the same proof, changing the constants. Denote with $l_j=l_0 2^j$. In order to prove the proposition we write the telescopic sum

$$\Psi_x^{l_0}(\sigma,\alpha^{-1}s)-\Psi_x^{l}(\sigma,\alpha^{-1}s)=\sum_{j=0}^{J-1}\Psi_x^{l_j}(\sigma,\alpha^{-1}s)-\Psi_x^{2l_j}(\sigma,\alpha^{-1}s)\ .$$

An application of the Minkovski inequality shows that

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{0\leq t\leq T}\left(\int_0^t \epsilon^d \sum_{x\in\Lambda_N} \nabla_\epsilon\,\phi(\epsilon x,s)\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,\alpha^{-1}s)\right)\right.\right.$$
$$\left.\left.\times\,\delta^{-2}\left(\Psi_x^{l_0}(\sigma,\alpha^{-1}s)-\Psi_x^{l}(\sigma,\alpha^{-1}s)\right)ds\right)^2\right]^{\frac{1}{2}}$$
$$\lesssim\sum_{j=0}^{J-1}\alpha^{\frac{1}{2}}\epsilon^{\frac{d}{2}}l_j^{1-\frac{d}{2}}\delta^{-2}\left(\int_0^T \mathbb{E}_{\mu_\gamma}\left\|\nabla_\epsilon\,\phi\,\nabla_\epsilon\,X_\gamma\right\|^2_{L^2(\Lambda_\varepsilon)}(s)\,ds\right)^{\frac{1}{2}}\ .$$

Summing over $j=0,\cdots,J-1$ we obtain

$$\sum_{j=0}^{J-1}2^{j\frac{-d+2}{2}}\lesssim\begin{cases}2^{\frac{J}{2}} & \text{if } d=1\\ J & \text{if } d=2\end{cases}$$

and this yield the result in (4.85). □

Propositions 4.6.12, 4.6.14 show that for $d=1$ we can replace the function $\sigma_x\sigma_{x+1}$

for $l \lesssim \gamma^{-1}$ with $\Psi^l_x(\sigma)$ at cost

$$\gamma^{\frac{10}{3}} l \int_0^T \left\| \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma \right\|^2_{L^2(\Lambda_\varepsilon)} |\nabla_\epsilon X_\gamma(x,s)|^2 ds \lesssim \gamma^2 l \lesssim \gamma$$

Where the last expectation has been estimated using the regularity of κ_γ and the deterministic bound

$$|\nabla_\epsilon X_\gamma(x,s)| \lesssim \epsilon^{-1}\delta^{-1}\gamma \sim \gamma^{-\frac{2}{3}}$$

In order to complete the replacement, we need to prove that

$$\Psi^l_x(\sigma) \simeq (1+\mathfrak{g}_\gamma) h_\gamma(x) h_\gamma(x+1) - \mathfrak{g}_\gamma \; . \tag{4.86}$$

The replacement (4.86) is more difficult to obtain with technique used in Proposition (4.6.14) because $l \sim \gamma^{-1}$ is the scale of the interaction between the spins.

We will now provide some evidence to convince of the validity of the replacement (4.86). Assume for simplicity that, instead of $\Psi^l_x(\sigma)$ we had

$$\mathrm{Av}_{i \neq j \in B^l_x} \sigma_i \sigma_j$$

and that, instead of $(1+\mathfrak{g}_\gamma) h_\gamma(x) h_\gamma(x+1) - \mathfrak{g}_\gamma$ we had

$$\mathrm{Av}_{i \neq j \in B^l_x} \mu_\gamma[\sigma_i \sigma_j | \mathcal{F}_{\{i,j\}^c}] \; .$$

Those substitutions are quite natural, as we will show in Lemmas 4.6.17 and 4.6.18. The next proposition corresponds to [GJ13a, Proposition 3.2] and exploits the particular property of the Gibbs measure, namely

$$\mu_\gamma[\sigma_x | \mathcal{F}_{\{x\}^c}] = \mu_\gamma[\tanh(\beta h_\gamma(x)) | \mathcal{F}_{\{x\}^c}]$$

and not the transport property of the Kawasaki dynamic. This is essentially inefficient because of the long time regime we are interested in. Indeed the following proposition would not be helpful any more in dimension two. The proposition show that the replacement (4.86) can be performed in $L^2([0,T] \times \Lambda_\varepsilon)$ for $l \gg \gamma^{\frac{2}{3}}$. A possible generalization to any $L^p([0,T] \times \Lambda_\varepsilon)$ might yield a proof of replacement (4.86) in the norm $L^\infty([0,T] \times \Lambda_\varepsilon)$. Recall the notation

$$h^{\{x_1,x_2\}}_\gamma(x) = \sum_{z \in \Lambda_N \setminus \{x_1,x_2\}} \kappa_\gamma(x,z) \sigma_z \; .$$

where we excluded the sites $\{x_1, x_2\}$ in the sum.

Proposition 4.6.15 *If for all $q \geq 1$ and $\kappa > 0$*

$$limsup_{\gamma\to 0}\gamma^{\kappa}\mu_\gamma\left[\|X_\gamma(s)\|^q_{L^\infty(\Lambda_\varepsilon)}\right] < \infty$$
$$limsup_{\gamma\to 0}\gamma^{\frac{q}{6}+\kappa}\mu_\gamma\left[\|\nabla_\epsilon X_\gamma(s)\|^q_{L^\infty(\Lambda_\varepsilon)}\right] < \infty$$

Then, for $l \leq \mathbf{a}\gamma^{-1}$, there exists $C = C(\kappa)$ such that

$$\mathbb{E}_{\mu_\gamma}\left[\sup_{t\leq T}\left(\int_0^t \epsilon\sum_{x\in\Lambda_N}\nabla_\epsilon\,\phi(\epsilon x,s)\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,\alpha^{-1}s)\right)\right.\right.$$
$$\left.\left.\times\,\delta^{-2}\left(\mathrm{Av}_{i_1\neq j_1\in B_x^l}\,\sigma_{i_1}\sigma_{j_1}-\mu_\gamma[\sigma_{i_1}\sigma_{j_1}|\mathcal{F}_{\{i_1,j_1\}^c}]\right)ds\right)^2\right]$$
$$\leq C\left(\gamma^{\frac{1}{3}-\kappa}+\gamma^{-\frac{2}{3}-\kappa}l^{-1}+\gamma^{-\frac{4}{3}-\kappa}l^{-2}\right)T\int_0^T\|\nabla_\epsilon\,\phi(s)\|^2_{L^2(\Lambda_\varepsilon)}\,ds \quad (4.87)$$

Proof. Using Cauchy-Schwarz we bound the left-hand-side of (4.87) with

$$T\int_0^T\epsilon^2\sum_{x,y\in\Lambda_N}|\nabla_\epsilon\,\phi(\epsilon x,s)\,\nabla_\epsilon\,\phi(\epsilon y,s)|ds$$
$$\times\,\gamma^{-\frac{14}{3}}\,\mathrm{Av}_{\substack{i_1\neq j_1\in B_x^l\\ i_2\neq j_2\in B_y^l}}\,\mu_\gamma\Big[\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x)\right)\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(y)\right)$$
$$\times\left(\sigma_{i_1}\sigma_{j_1}-\mu_\gamma[\sigma_{i_1}\sigma_{j_1}|\mathcal{F}_{\{i_1,j_1\}^c}]\right)\left(\sigma_{i_2}\sigma_{j_2}-\mu_\gamma[\sigma_{i_2}\sigma_{j_2}|\mathcal{F}_{\{i_2,j_2\}^c}]\right)\Big] \quad (4.88)$$

where we used the fact that the Gibbs measure μ_γ is stationary for the dynamic. The factor $\gamma^{-\frac{14}{3}}$ comes from $\delta^{-6}\epsilon^{-2}$ and (4.20).
From the expression of $\mu_\gamma[\sigma_{i_1}\sigma_{j_1}|\mathcal{F}_{\{i_1,j_1\}^c}]$ provided in Lemma 4.6.18 we see that if $|x-y|\geq\mathrm{diam}(\kappa_\gamma)+2l$ the quantity inside the summation vanishes. If $|x-y|\leq\mathrm{diam}(\kappa_\gamma)+2l$ we can use the fact that $\sigma_{i_1}\sigma_{j_1}-\mu_\gamma[\sigma_{i_1}\sigma_{j_1}|\mathcal{F}_{\{i_1,j_1\}^c}]$ has mean zero with respect to $\mu_\gamma[\cdot|\mathcal{F}_{\{i_1,j_1\}^c}]$ provided it is multiplied with a quantity measurable with respect to $\mathcal{F}_{\{i_1,j_1\}^c}$.
We will now compute the expectation in (4.88). It is convenient to use the following

$$\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(y)\right)=\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma^{\{i_1,j_1\}}(y)\right) \quad (4.89)$$
$$+\,\beta\tanh'\left(\beta\nabla_N^+\tilde{h}_\gamma^{\{i_1,j_1\}}(y)\right)(\nabla_N^+\kappa_\gamma(y,i_1)\sigma_{i_1}+\nabla_N^+\kappa_\gamma(y,j_1)\sigma_{j_1})+\mathcal{O}(\gamma^4)$$
$$\mu_\gamma[\sigma_{i_2}\sigma_{j_2}|\mathcal{F}_{\{i_2,j_2\}^c}]=\mu_\gamma[\sigma_{i_2}\sigma_{j_2}|\mathcal{F}_{\{i_2,j_2,i_1,j_1\}^c}]+\mathcal{O}(\gamma h_\gamma(i_2)+\gamma h_\gamma(j_2)+\gamma^2) \quad (4.90)$$

which are a consequence of the Taylor expansion of the hyperbolic tangent and Lemma 4.6.18 respectively. Recall moreover that $\nabla_N^+\tilde{h}_\gamma(y)$ is measurable with respect to $\mathcal{F}_{B_y^l}$ for $l\leq\mathbf{a}\gamma^{-1}$ because of our assumption on the kernel.

We decompose the expectation in (4.88), using (4.89), into the sum

$$\sum_{k=1}^{5} D_k(i_1, j_1, i_2, j_2)$$

where

$$\begin{aligned} D_1(i_1, j_1, i_2, j_2) = \mu_\gamma \Big[& \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_2,j_2\}}(x)\right) \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_1,j_1\}}(y)\right) \\ & \times \left(\sigma_{i_1}\sigma_{j_1} - \mu_\gamma[\sigma_{i_1}\sigma_{j_1} | \mathcal{F}_{\{i_1,j_1\}^c}]\right) \left(\sigma_{i_2}\sigma_{j_2} - \mu_\gamma[\sigma_{i_2}\sigma_{j_2} | \mathcal{F}_{\{i_2,j_2\}^c}]\right) \Big] \end{aligned}$$

$$\begin{aligned} D_2(i_1, j_1, i_2, j_2) = \mu_\gamma \Big[& \tanh\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_2,j_2\}}(x)\right) \beta \tanh'\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_1,j_1\}}(y)\right) \\ & \times \Big\{ \nabla_N^+ \kappa_\gamma(y, i_1)[\sigma_{j_1} - \sigma_{i_1}\mu_\gamma[\sigma_{i_1}\sigma_{j_1} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \\ & + \nabla_N^+ \kappa_\gamma(y, j_1)[\sigma_{i_1} - \sigma_{j_1}\mu_\gamma[\sigma_{i_1}\sigma_{j_1} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \Big\} \\ & \times \left(\sigma_{i_2}\sigma_{j_2} - \mu_\gamma[\sigma_{i_2}\sigma_{j_2} | \mathcal{F}_{\{i_2,j_2\}^c}]\right) \Big] \end{aligned}$$

and similarly for D_3, while

$$\begin{aligned} D_4(i_1, j_1, i_2, j_2) = \mu_\gamma \Big[& \beta \tanh'\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_2,j_2\}}(x)\right) \beta \tanh'\left(\beta \nabla_N^+ \tilde{h}_\gamma^{\{i_1,j_1\}}(y)\right) \\ & \times \Big\{ \nabla_N^+ \kappa_\gamma(y, i_1)[\sigma_{j_1} - \sigma_{i_1}\mu_\gamma[\sigma_{i_1}\sigma_{j_1} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \\ & + \nabla_N^+ \kappa_\gamma(y, j_1)[\sigma_{i_1} - \sigma_{j_1}\mu_\gamma[\sigma_{i_1}\sigma_{j_1} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \Big\} \\ & \times \Big\{ \nabla_N^+ \kappa_\gamma(x, i_2)[\sigma_{j_2} - \sigma_{i_2}\mu_\gamma[\sigma_{i_2}\sigma_{j_2} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \\ & + \nabla_N^+ \kappa_\gamma(x, j_2)[\sigma_{i_2} - \sigma_{j_2}\mu_\gamma[\sigma_{i_2}\sigma_{j_2} | \mathcal{F}_{\{i_1,j_1,i_2,j_2\}^c}]] \Big\} \Big] \end{aligned}$$

and finally, using (4.89) and (4.90)

$$\begin{aligned} & \mu_\gamma[D_5(i_1, j_1, i_2, j_2)] \\ & \qquad \lesssim \gamma^{\frac{17}{3}} \mu_\gamma[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)}] + \gamma^{\frac{15}{3}} \mu_\gamma[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)}] \end{aligned}$$

In case $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, using (4.89), (4.90), Lemma 4.6.18 we have

$$\mu_\gamma[D_1(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{15}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} \left(\|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} + \gamma \right)^2 \right]$$
$$\mu_\gamma[D_2(i_1, j_1, i_2, j_2)] = \mu_\gamma[D_2(i_1, j_1, i_2, j_2)] = 0$$
$$\mu_\gamma[D_4(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{14}{3}} \mu_\gamma \left[\|X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} \right]$$

Otherwise if $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$

$$\mu_\gamma[D_1(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{12}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} \left(\|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} + \gamma \right)^2 \right]$$
$$\mu_\gamma[D_2(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{12}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \right]$$
$$\mu_\gamma[D_3(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{12}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \right]$$
$$\mu_\gamma[D_4(i_1, j_1, i_2, j_2)] \lesssim \gamma^4$$

Finally if $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 2$

$$\mu_\gamma[D_1(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{10}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} + \gamma^2 \right]$$
$$\mu_\gamma[D_2(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{12}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \right]$$
$$\mu_\gamma[D_3(i_1, j_1, i_2, j_2)] \lesssim \gamma^{\frac{12}{3}} \mu_\gamma \left[\|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \right]$$
$$\mu_\gamma[D_4(i_1, j_1, i_2, j_2)] \lesssim \gamma^4$$

With the calculations above we have that (4.88) is bounded by

$$\begin{aligned} T \int_0^T \|\nabla_\epsilon \phi(s)\|^2_{L^2(\Lambda_\varepsilon)} \, ds \mu_\gamma \Big[(\gamma^{\frac{2}{3}} + \gamma^{-\frac{1}{3}} l^{-1}) \|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} \|X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} \\ + \gamma^{\frac{1}{3}} \|X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} + \gamma^{-\frac{1}{3}} l^{-1} \|X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \|\nabla_\epsilon X_\gamma\|_{L^\infty(\Lambda_\varepsilon)} \\ + \gamma^{-1} l^{-2} \|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)} + \gamma^{-\frac{1}{3}} l^{-2} \Big] \end{aligned}$$

and the result follows from the assumptions over $\mu_\gamma[\|\nabla_\epsilon X_\gamma\|^2_{L^\infty(\Lambda_\varepsilon)}]$. □

Remark 4.6.16 The hypothesis of Proposition 4.6.15 are quite natural and indeed we expect to prove them in one dimension in the same way as the proof of Theorem 3.2.1 in Chapter 3.

The next lemma provides a description of $\Psi^l_x(\sigma)$ in term of $\bar{\sigma}^l_x$. Let $\eta \in \{-1, 1\}^{\Lambda_N}$

and $M \in \{-1, -1+2/|\Lambda|, \cdots, 1-2/|\Lambda|, 1\}$. Recall the definitions of the Hamiltonian

$$\mathcal{H}_\gamma^{\Lambda,\eta}(\sigma) = \frac{\beta}{2} \sum_{x,y \in \Lambda} \kappa_\gamma(x,y)\sigma_x\sigma_y + \beta \sum_{x \in \Lambda, y \notin \Lambda} \kappa_\gamma(x,y)\sigma_x\eta_y$$

and, respectively, the Grand Canonical and Canonical Gibbs measure over the domain Λ

$$\mu_\gamma^{\Lambda,\eta}(\sigma) = \left(Z_\gamma^{\Lambda,\eta}\right)^{-1} \exp\left\{\mathcal{H}_\gamma^{\Lambda,\eta}(\sigma)\right\}$$
$$\mu_\gamma^{\Lambda,\eta,M}(\sigma) = \left(Z_\gamma^{\Lambda,\eta,M}\right)^{-1} \exp\left\{\mathcal{H}_\gamma^{\Lambda,\eta}(\sigma)\right\} 1_{\{\mathrm{Av}_{i\in\Lambda}\,\sigma_i = M\}}\,.$$

Lemma 4.6.17 *Let Λ be a block of diameter l centered in the origin, and let $l \le \mathbf{a}\gamma^{-1}$, where $\mathbf{a}$ is the constant of* (FLAT)*. We have*

$$\mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1] = -\frac{1}{|\Lambda|-1} + \frac{|\Lambda|}{|\Lambda|-1}\left\{M + \mathcal{O}\big(\mathrm{Av}_{i\in\Lambda}|\tilde\kappa_\gamma * \eta(i) - \tilde\kappa_\gamma * \eta(0)|\big)\right\}^2$$
$$+ \mathcal{O}\left(l^{-d} + \mathrm{Av}_{i\in\Lambda}|\tilde\kappa_\gamma * \eta(i) - \tilde\kappa_\gamma * \eta(0)|\right)^2$$

Proof. The proof is an easy consequence of Remark 4.6.7. In order to keep the formulas contained, we use $p_y \stackrel{\text{def}}{=} \beta\tilde\kappa_\gamma * \eta(x)$. We have

$$\mu_\gamma^{\Lambda,\eta,M}[\sigma_0] = M + \frac{1}{|\Lambda|}\sum_{i\in B\setminus\{0\}} \tanh(p_0 - p_i)(1 - \mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_i])$$

$$\begin{aligned}
\mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1] &= -\frac{1}{|\Lambda|-1} + \frac{|\Lambda|}{|\Lambda|-1}|M|\mu_\gamma^{\Lambda,\eta,M}[\sigma_0] \\
&+ \frac{1}{|\Lambda|-1}\sum_{i\in B\setminus\{0,1\}} \tanh(p_1 - p_i)\mu_\gamma^{\Lambda,\eta,M}[\sigma_0(1-\sigma_1\sigma_i)] \\
&= -\frac{1}{|\Lambda|-1} + \frac{|\Lambda|}{|\Lambda|-1}\mu_\gamma^{\Lambda,\eta,M}[\sigma_0]^2 + O(l^{-d}\gamma^2) + \\
&+ \frac{1}{|\Lambda|-1}\sum_{i\in B\setminus\{0,1\}} \tanh(p_1 - p_i)\left(\mu_\gamma^{\Lambda,\eta,M}[\sigma_0]\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i] - \mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1\sigma_i]\right)
\end{aligned}$$

In order to estimate the last term $\mu_\gamma^{\Lambda,\eta,M}[\sigma_0]\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i]-\mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1\sigma_i]$ we also calculate

$$\begin{aligned}\mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1\sigma_i] &= M\frac{|\Lambda|}{|\Lambda|-2}\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i] \\ &\quad - \frac{1}{|\Lambda|-2}\sum_{j\in\Lambda\setminus\{1,i\}}\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i(1-\sigma_0\sigma_j)]\tanh(p_j-p_0) \\ &= \mu_\gamma^{\Lambda,\eta,M}[\sigma_0]\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i]+\mathcal{O}\left(l^{-d}+\mathrm{Av}_{j\in B}\,|p_j-p_0|\right)\end{aligned}$$

and therefore, uniformly for $i\in\lambda$ we have

$$\left|\mu_\gamma^{\Lambda,\eta,M}[\sigma_0\sigma_1\sigma_i]-\mu_\gamma^{\Lambda,\eta,M}[\sigma_0]\mu_\gamma^{\Lambda,\eta,M}[\sigma_1\sigma_i]\right|\lesssim l^{-d}+\mathrm{Av}_{j\in B}\,|p_j-p_0|\ .$$

Using the previous bound we obtain the proposition. □

The next lemma contains some estimates used in the proof of Proposition 4.6.15. The proof follows from the fact that, for any finite set of spins of cardinality $n\in\mathbb{N}$, the Grand Canonical Gibbs measure, restricted to such set is approximately a product measure, if $n\ll\gamma^{-1}$ (see also Remark 4.6.7).

Lemma 4.6.18 *Let*

$$\overline{\sigma_{x_1}\sigma_{x_2}}\stackrel{def}{=}\sigma_{x_1}\sigma_{x_2}-\mu_\gamma\left[\sigma_{x_1}\sigma_{x_2}\middle|\mathcal{F}_{\{x_1,x_2\}^c}\right]$$

Then, for different $x_1,x_2\in\Lambda_N$ we have

$$\begin{aligned}&\mu_\gamma\left[\sigma_{x_1}\sigma_{x_2}\middle|\mathcal{F}_{\{x_1,x_2\}^c}\right] \\ &\qquad = \frac{\tanh(\beta h_\gamma^{\{x_1,x_2\}}(x_1))\tanh(\beta h_\gamma^{\{x_1,x_2\}}(x_2))+\tanh(\beta\kappa_\gamma(x_1,x_2))}{1+\tanh(\beta h_\gamma^{\{x_1,x_2\}}(x_1))\tanh(\beta h_\gamma^{\{x_1,x_2\}}(x_2))\tanh(\beta\kappa_\gamma(x_1,x_2))}\end{aligned}$$

Moreover for different $x_1,x_2,y_1,y_2\in\Lambda_N$

$$\begin{aligned}&\mu_\gamma\left[\overline{\sigma_{x_1}\sigma_{x_2}}\quad\overline{\sigma_{y_1}\sigma_{y_2}}\middle|\mathcal{F}_{\{x_1,x_2,y_1,y_2\}^c}\right] \\ &\qquad\lesssim\left(\|h_\gamma\|^2_{L^\infty(\Lambda_N)}+\gamma\right)\left(\kappa_\gamma(x_1,y_1)+\kappa_\gamma(x_2,y_1)+\kappa_\gamma(x_1,y_2)+\kappa_\gamma(x_2,y_2)\right)\end{aligned}$$

In the next section we are going to provide more arguments towards the replacement in (4.86), in a more general setting.

4.6.4 Nonequilibrium fluctuation

The aim of this section is to extend the previous subsection to the case when the Kawasaki process is started from an arbitrary initial condition. The idea is classic and has been inspired by the works [Qua96, Rez94] and [KOV89]. We will measure the distance between a generic measure π over Σ_N and the Gibbs measure μ_γ with the entropy

$$H(\pi/\mu_\gamma) = \mathbb{E}_{\mu_\gamma}\left[\frac{d\pi}{d\mu_\gamma}\log\left(\frac{d\pi}{d\mu_\gamma}\right)(\sigma)\right] \leq C\epsilon^{-d}$$

since the Hamiltonian (4.1) is bounded by a constant times ϵ^{-d} uniformly in the configuration.

Moreover, for any $T > 0$, the Radon-Nikodym derivative between the probability measures $\mathbb{P}_\pi$ and $\mathbb{P}_\pi$ on $\mathcal{D}([0,T];\Sigma_N)$ given by the Kawasaki processes started, respectively from the measure π and from the invariant measure μ_γ, only depends on the Radon-Nikodym derivative between π and μ_γ at the initial condition (see [KL99, App 1]). Therefore the entropy between $\mathbb{P}_\pi$ and $\mathbb{P}_{\mu_\gamma}$ satisfies

$$H(\mathbb{P}_\pi/\mathbb{P}_{\mu_\gamma}) = H(\pi/\mu_\gamma) \lesssim \epsilon^{-d}\,.$$

The importance of the entropy is given by the following inequalities. For two probability measure p_1 and p_2 on the same space $(\Omega, \mathcal{A})$ with finite relative entropy $H(p_1/p_2) < \infty$, we have the following inequality [KL99, Appendiix 1, Proposition 8.2] for any event $A \in \mathcal{A}$

$$p_2(A) \leq \frac{\log(2) + H(p_2/p_1)}{\log(1 + p_1(A)^{-1})}$$

This is particular case of the more general inequality

$$\mathbb{E}_{p_2}[X] \leq H(p_2/p_1) + \log\mathbb{E}_{p_1}[\exp\{X\}]\,. \tag{4.91}$$

Let $\phi : \Lambda_\varepsilon \times [0,T] \to \mathbb{R}$ be a test function, as in Subsection 4.6.3. To simplify some notations we ignore the higher power in the Taylor expansion of the hyperbolic tangent

$$\delta^{-1}\epsilon^{-1}\tanh\left(\beta\nabla_N^+\tilde{h}_\gamma(x,\alpha^{-1}s)\right) = \nabla_\epsilon\,\tilde{X}_\gamma(x,s) + \delta^2\epsilon^2\mathcal{O}\left(|\,\nabla_\epsilon\,\tilde{X}_\gamma(x,s)|^3\right)$$

one can immediately see, using the scaling (4.20) and the naive bound $|\nabla_N^+\tilde{h}_\gamma| \lesssim \gamma$, that the error is negligible if $d = 1$.

The next proposition correspond to Proposition 4.6.12, reproven when the Kawasaki process is started a generic measure π over the space of configurations Σ_N.

Proposition 4.6.19 (One block estimate) *Let $0 < l \leq \mathbf{a}\gamma^{-1}$, and consider a family of functions $f_{x,l} : \Sigma_N \to \mathbb{R}$ for $x \in \Lambda_\varepsilon$, having zero mean with respect to any Canonical Gibbs measure in the block B_x^l*

$$\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[f_{x,l}] = 0$$

and satisfying $|f_{x,l}(\sigma)| \leq 1$. Then, for all $\Gamma > 0$

$$\begin{aligned}
&\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \sum_{x\in\Lambda_\varepsilon} \epsilon^d \nabla_\epsilon \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s))\nabla_\epsilon \tilde{X}_\gamma(x,s)ds\right|\right] \\
&\quad \leq \Gamma^{-1}\left(H(\pi/\mu_\gamma) + 2d\log(\epsilon^{-1})\right) \qquad (4.92)\\
&\quad + C_0\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\int_0^t \sum_{x\in\Lambda_\varepsilon}\epsilon^d|\nabla_\epsilon \phi(x-z,s)|^2|\nabla_\epsilon \tilde{X}_\gamma(x,s)|^2 ds\right]
\end{aligned}$$

Remark 4.6.20 In case we start from the measure $\pi = \mu_\gamma$, using (4.92) and optimizing over Γ we would have obtained a bound of the form

$$\begin{aligned}
&\mathbb{E}_{\mu_\gamma}\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \sum_{x\in\Lambda_\varepsilon} \epsilon^d \nabla_\epsilon \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s))\nabla_\epsilon \tilde{X}_\gamma(x,s)ds\right|\right] \\
&\quad \lesssim \epsilon^{d/2}\log^{1/2}(\epsilon^{-1})\alpha^{1/2}\delta^{-2}l^{d/2+1} \\
&\quad \times \mathbb{E}_{\mu_\gamma}\left[\int_0^t \|\nabla_\epsilon \phi(\cdot,s)\|^2_{L^2(\Lambda_\varepsilon)}\left\|\nabla_\epsilon \tilde{X}_\gamma(\cdot,s)\right\|^2_{L^\infty(\Lambda_\varepsilon)} ds\right]^{1/2}
\end{aligned}$$

which is indeed essentially the statement of Proposition 4.6.12.

As discussed in the previous section, the one block estimate becomes inefficient when the size of the block grows. The strategy used in Subsection 4.6.3, namely to double the diameter of the block until it reaches a mesoscopic size, relies heavily on the form of the grand canonical Gibbs measures $\mu_\gamma^{\Lambda,\eta}$. However, looking at the proof Proposition 4.6.19, we can prove still prove the following corollary.

Corollary 4.6.21 *Recall the definition* (4.78). *In the same setting of Proposition 4.6.19, we have*

$$\begin{aligned}
&\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \sum_{x\in\Lambda_\varepsilon} \epsilon^2 \nabla_\epsilon \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s))\nabla_\epsilon \tilde{X}_\gamma(x,s)ds\right|\right] \\
&\quad \leq \epsilon^{d/2}\alpha^{1/2}l^{d/2}\delta^{-2}\left(H(\pi/\mu_\gamma) + 2d\log(\epsilon^{-1})\right)^{\frac{1}{2}} \qquad (4.93)\\
&\quad \times \mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\int_0^t \sum_{x\in\Lambda_\varepsilon}\epsilon^2|\nabla_\epsilon \phi(x-z,s)|^2\mathbf{V}_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}(f_{x,l})|\nabla_\epsilon \tilde{X}_\gamma(x,s)|^2 ds\right]^{1/2}
\end{aligned}$$

The result (4.93) might turn useful, as soon as one gains some control over

$$\mathbf{V}_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}(f) = \sup_{g \text{ local}} \left\{ 2\, \mu_\gamma^{\Lambda,\eta,M}\left[f\, g\right] - \mu_\gamma^{\Lambda,\eta,M}\left[g(-\mathscr{L}_{\Lambda,\eta}^K g)\right]\right\} \tag{4.94}$$

Quantities of this form arise in the computation of the diffusion coefficient in the context of hydrodynamic limits. In Chapter 7, Theorem 4.1 of [KL99] it is proven that (4.94) remains bounded as $\gamma \to 0$, in case the expectation is taken wih respect to a product measure and the process is the symmetric generalized simple exclusion process. The same result has been proven in more general setting (see [VY97]) and it is therefore natural to conjecture the boundedness of (4.94) also in the case of Kawasaki dynamic introduced in this chapter. The approach in [VY97] doesn't seem to cover the case of a Gibbs measure with Kac interaction, since the estimates are not uniform in the range of the interaction. However, due to the flatness of the kernel, the result might still hold true for $l \ll \gamma^{-1}$, which is the regime we are interested in. Since this is not needed for the rest of the argument, and we don't have a convenient bound for (4.94), we are not going to benefit from the more precise formulation in (4.93).

Proof of Proposition 4.6.19. We will denote, for convenience

$$\begin{aligned}
\psi_1(z,s) &= \sum_{x\in\Lambda_\varepsilon} \epsilon^d\, \nabla_\epsilon\, \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s))\, \nabla_\epsilon\, \tilde{X}_\gamma(x,s)\\
\psi_2(z,s) &= \sum_{x\in\Lambda_\varepsilon} \epsilon^d |\, \nabla_\epsilon\, \phi(x-z,s)|^2 |\, \nabla_\epsilon\, \tilde{X}_\gamma(x,s)|^2\ .
\end{aligned}$$

With the above notations, for all $\theta > 0$ we have

$$\begin{aligned}
\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \psi_1(z,s)ds\right|\right] &\le \mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\theta\int_0^t \psi_2(z,s)\, ds\right]\\
&\quad + \mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left\{\left|\int_0^t \psi_1(z,s)ds\right| - \theta\int_0^t \psi_2(z,s)ds\right\}\right]
\end{aligned}$$

We will now proceed to bound the last expectation for a suitable value of θ. Using the entropy inequality, the above quantity is bounded by

$$\Gamma^{-1}H(\pi/\mu_\gamma) + \Gamma^{-1}\log \mathbb{E}_{\mu_\gamma}\left[\exp\left(\Gamma\sup_{z\in\Lambda_\varepsilon}\left\{\left|\int_0^t \psi_1(z,s)ds\right| - \theta\int_0^t \psi_2(z,s)ds\right\}\right)\right]$$

Moreover, we can now bound the supremum with the sum over all z and use the elementary inequality $\exp(|a|) \leq \exp(a) + \exp(-a)$

$$\log \mathbb{E}_{\mu_\gamma} \exp\left(\Gamma \sup_{z\in\Lambda_\varepsilon}\left\{\left|\int_0^t \psi_1(z,s)ds\right| - \theta\int_0^t \psi_2(z,s)ds\right\}\right)$$
$$\leq 2d\log(\epsilon^{-1}) + \log\sum_{z\in\Lambda_\varepsilon} \epsilon^d \mathbb{E}_{\mu_\gamma} \exp\left(\Gamma\int_0^t \psi_1(z,s)ds - \theta\psi_2(z,s)\,ds\right)$$
$$+\log\sum_{z\in\Lambda_\varepsilon} \epsilon^d \mathbb{E}_{\mu_\gamma} \exp\left(\Gamma\int_0^t -\psi_1(z,s)\,ds - \theta\psi_2(z,s)ds\right). \quad (4.95)$$

By Lemma 4.6.22 we have that the last expectations are negative if $\theta = C_0\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}$ and therefore

$$E_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \psi_1(z,s)ds\right|\right]$$
$$\leq \Gamma^{-1}\left(H(\pi/\mu_\gamma) + 2d\log(\epsilon^{-1})\right) + C_0\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\int_0^t \psi_2(z,s)\,ds\right]$$

and optimizing over Γ we obtain the statement. □

We will now prove the main estimate used in the previous proposition.

Lemma 4.6.22 *Let ψ_1, ψ_2 be defined as in the proof of 4.6.19. Then there exists $C_0 > 0$ such that for all $\Gamma > 0$, and for $\theta = C_0\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}$*

$$\log \mathbb{E}_{\mu_\gamma}\left[\exp\left(\Gamma\int_0^t \psi_1(z,s)ds - \theta\psi_2(z,s)\,ds\right)\right] \leq 0 \quad (4.96)$$

Proof. We bound the above expectations with the Feynman-Kac formula [KL99, Appendiix 1, Lemma 7.2]

$$\log \mathbb{E}_{\mu_\gamma} \exp\left(\Gamma\left\{\int_0^t \pm\psi_1(z,s)\,ds - \theta\psi_2(z,s)ds\right\}\right)$$
$$\leq \int_0^t \sup\operatorname{spec}_{L^2(\mu_\gamma)}\left\{\pm\Gamma\psi_1(z,s) - \theta\Gamma\psi_2(z,s) + \alpha^{-1}\mathscr{L}^K\right\}ds$$

where the quantity inside the integral denotes the largest eigenvalue of the operator

$$\pm\Gamma\psi_1(z,s) - \theta\Gamma\psi_2(z,s) + \alpha^{-1}\mathscr{L}^K$$

and satisfies the variational formula

$$\sup_{g:\mu_\gamma[g^2]=1} \mu_\gamma \left[g(\sigma) \left(\pm\Gamma\psi_1(z,s) - \theta\Gamma\psi_2(z,s) + \alpha^{-1}\mathscr{L}^K \right) g(\sigma) \right] . \tag{4.97}$$

and using the definitions of ϕ_1, ϕ_2, (4.97) is bounded by

$$\sup_{g:\mu_\gamma[g^2]=1} \mu_\gamma \Bigg[\sum_{x\in\Lambda_\varepsilon} \Gamma\epsilon^d \nabla_\epsilon \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s)) \nabla_\epsilon \tilde{X}_\gamma(x,s) g^2(\sigma)$$
$$- \Gamma\theta\epsilon^d |\nabla_\epsilon \phi(x-z,s)|^2 |\nabla_\epsilon \tilde{X}_\gamma(x,s)|^2 g^2(\sigma) \Bigg] - \alpha^{-1}\mathbf{D}_\gamma^{\Lambda_N}[g] . \tag{4.98}$$

We now bound the above sum termwise. Notice at first that, for each x in the sum, using $|f_{x,l}(\sigma)| \leq 2$, we can assume

$$|\nabla_\epsilon \phi(x-z,s)\delta^{-2} f_{x,l}(\sigma(\alpha^{-1}s)) \nabla_\epsilon \tilde{X}_\gamma(x,s)| \leq \frac{1}{2}\delta^{-2}\theta^{-1} \tag{4.99}$$

to hold. Indeed, it is easy to see that if (4.99) doesn't hold, the corresponding term in (4.98) is negative and we have nothing to prove. Recall that $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[f_x] = 0$ and $\nabla_\epsilon \tilde{X}_\gamma(x,s)$ depends only on the spins outside a ball of radius $\mathbf{a}\gamma^{-1}$, and therefore for any $\eta \in \Sigma_N$ and $M \in [-1,1]$ it is $\mu_\gamma^{B_x^l,\eta,M}$-a.s. constant.
Moreover, by an easy combinatorial argument over the bonds of Λ_N,

$$\mu_\gamma \left[g(\sigma)(-\mathscr{L}^K) g(\sigma) \right] \geq c_0^{-1} l^{-d} \mu_\gamma \sum_{x\in\Lambda_N} \mathbf{D}_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[g]$$

We bound the the sum (4.98) using the spectrum calculated with respect to $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$. In order to do so let us introduce

$$\bar{g}_x^l(\sigma) := \frac{g(\sigma)}{\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[g^2]^{1/2}} .$$

From the definition $\bar{g}_x^l(\sigma)$ has square mean 1 with respect to the canonical measure $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$. Since $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[g^2]^{1/2}$ is $\mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}$-a.s. constant, we can write

$$\mathbf{D}_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[g] = \mu_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[g^2] \mathbf{D}_\gamma^{B_x^l,\sigma,\bar{\sigma}_x^l}[\bar{g}_x^l]$$

then, (4.98) is equal to

$$\sup_{g:\mu_\gamma[g^2]=1} \mu_\gamma \sum_{x\in\Lambda_\varepsilon} \mu_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}[g^2] c_0^{-1}\alpha^{-1} l^{-d}$$
$$\Bigg\{ c_0\alpha\epsilon^d l^d \delta^{-2}\Gamma \,\nabla_\epsilon\, \phi(x-z,s)\,\nabla_\epsilon\, \tilde X_\gamma(x,s)\mu_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}\Big[f_{x,l}(\sigma(\alpha^{-1}s))(\bar g^l_x)^2(\sigma)\Big]$$
$$- c_0\alpha l^d\Gamma\epsilon^d\theta |\,\nabla_\epsilon\,\phi(x-z,s)|^2|\,\nabla_\epsilon\,\tilde X_\gamma(x,s)|^2(\bar g^l_x)^2(\sigma) - \mathbf{D}_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}[\bar g^l_x]\Bigg\} . \quad (4.100)$$

In the above formula we used the fact that $\nabla_\epsilon\, \tilde X_\gamma(x,s)$ is $\mu_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}$ constant. Recall that if V is a bounded multiplicative operator with $\mu(V)=0$ and L is a negative semidefinite operator with spectral gap $sp(L)$ we have (see [KL99, Appendiix 3,Theorem 1.1] for a proof)

$$\sup \operatorname{spec}_{L^2(\mu)}\{V+L\} \le \frac{1}{1-2\,\|V\|_\infty\, sp(L)^{-1}}\mu\left[V(-L)^{-1}V\right] .$$

We will use the previous formula with $L=\mathscr{L}^K_{B^l_x,\sigma}$ and

$$V = c_0\Gamma\alpha\delta^{-2}l^d\,\nabla_\epsilon\,\phi(x-z,s)f_{x,l}(\sigma(\alpha^{-1}s))\,\nabla_\epsilon\,\tilde X_\gamma(x,s) .$$

Using (4.99) and Proposition 4.6.9, we are able to use the previous bound provided

$$\|V\|_\infty\, sp(L)^{-1} \le 2c_0 C\Gamma\theta^{-1}\epsilon^d\delta^{-4}l^{d+2}\alpha$$

is small enough. In this case we can bound (4.100) with

$$\Gamma\epsilon^d\mu_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}[g^2]|\,\nabla_\epsilon\,\phi(x-z,s)|^2|\,\nabla_\epsilon\,\tilde X_\gamma(x,s)|^2$$
$$\times\left(\frac{c_0\Gamma\epsilon^d\delta^{-4}\alpha l^d}{1-4c_0C\Gamma\epsilon^d\theta^{-1}\delta^{-4}l^{d+2}\alpha}\mathbf{V}_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}(f_{x,l})-\theta\right) . \quad (4.101)$$

where $\mathbf{V}_\gamma$ is defined in (4.78).
In particular, using (4.80), we can bound $\mathbf{V}_\gamma^{B^l_x,\sigma,\bar\sigma^l_x}(f_{x,l}) \le c_3 l^2$ and therefore there exist C_0 such that (4.101) is negative for $\theta = C_0\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}$. □

Corollary 4.6.23 *Let ψ_1,ψ_2 be defined as in the proof of 4.6.19. The same proof shows that, there exists a constant $C_0>0$ such that for all $a>0$ and $\Gamma>0$ the following inequality* *holds*

$$\log \mathbb{P}_{\mu_\gamma}\left(\sup_{z\in\Lambda_\varepsilon}\left\{\left|\int_0^t \psi_1(z,s)ds\right| - \Gamma C_0\epsilon^d\delta^{-4}\alpha l^{d+2}\int_0^t \psi_2(z,s)ds\right\} > a\right)$$
$$\leq -\Gamma a + 2d\log(\epsilon^{-1}) \quad (4.102)$$

The advantage of Proposition 4.6.19 is that it already comes with a with a supremum in space. From the same proof it is easy to see that if $J\in\mathbb{N}$ and $\psi_1^{(j)},\psi_2^{(j)}$ for $j=1,\dots,J$ is a sequence of functions satisfying the statement of Lemma 4.6.22, then

$$\mathbb{E}_\pi\left[\sup_{j=1,\dots,J}\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \psi_1(z,s)ds\right|\right]$$
$$\leq \Gamma^{-1}\big(H(\pi/\mu_\gamma)+2d\log(\epsilon^{-1})+\log(J)\big)+\mathbb{E}_\pi\left[\sup_{j=1,\dots,J}\sup_{z\in\Lambda_\varepsilon}\int_0^t \psi_2(z,s)\,ds\right] \quad (4.103)$$

We will use the above inequality to insert a supremum not only with respect to the space variable, but also over the time variable. In order to do so we will make use of Proposition B.1.6 proven in the appendix in dimension 1. It is easy to see however that the same proof applies also in the case $d=2$, and yields a similar result. For this reason we will state the result for both dimensions, but we will prove it only in case of $d=1$.

Proposition 4.6.24 *For any $\lambda>0$ there exists a constant $C=C(\lambda)$ such that for any $\Gamma>0$ and $l\leq \mathbf{a}\gamma^{-1}$, we have*

$$\mathbb{E}_\pi\Bigg[\sup_{z\in\Lambda_\varepsilon}\sup_{0\leq t\leq T}\Bigg|\int_0^t \epsilon^d\sum_{x\in\Lambda_\varepsilon}\nabla_\epsilon P^{K,\gamma}_{t-s}K_\gamma(x-z)$$
$$\times\,\delta^{-2}\left(\sigma_x(\alpha^{-1}s)\sigma_{x+1}(\alpha^{-1}s)-\Psi^l_x(\sigma(\alpha^{-1}s))\right)\nabla_\epsilon\tilde{X}_\gamma(x,s)\,ds\Bigg|\Bigg]$$
$$\leq C\gamma^\lambda+\Gamma^{-1}\Big(H(\pi/\mu_\gamma)+2d\log(\epsilon^{-1})+C\log(\gamma^{-1})\Big) \quad (4.104)$$
$$+\,C\Gamma\epsilon^d\delta^{-4}\alpha l^{d+2}\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\int_0^T\sum_{x\in\Lambda_\varepsilon}\epsilon^d|\nabla_\epsilon P^{K,\gamma}_{t-s}K_\gamma(x-z)|^2|\nabla_\epsilon\tilde{X}_\gamma(x,s)|^2ds\right]$$

Proof. Assume $d=1$. Let $J\in\mathbb{N}$ and consider the discretization of the time $t_j=jTJ^{-1}\in[0,T]$ for $j=0,\dots,J$ we are later going to choose J proportional to a power of γ^{-1}.

$$\phi^{(j)}(x,s)=1_{\{s<t_j\}}P^{K,\gamma}_{t_j-s}K_\gamma(x)$$

and $f_{x,l} = \sigma_x(\alpha^{-1}s)\sigma_{x+1}(\alpha^{-1}s) - \Psi^l_x(\sigma(\alpha^{-1}s))$. Using (4.103), it is easy to see that the result holds true if we restrict the supremum over the time in (4.104) to a supremum over the set $\{t_j\}_{j=0}^J$, modifying slightly the proof of Proposition 4.6.19. To complete the proof it is sufficient to provide a deterministic bound over

$$\epsilon^{-1}\delta^{-3}\gamma \sup_{t_j\le t\le t_{j+1}} \int_{t_j}^t \epsilon^d \sum_{x\in\Lambda_\varepsilon} \left|\nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma(x)\right| ds$$
$$+ \epsilon^{-1}\delta^{-3}\gamma \sup_{t_j\le t\le t_{j+1}} \int_0^{t_j} \epsilon^d \sum_{x\in\Lambda_\varepsilon} \left|\nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma(x) - \nabla_\epsilon P^{K,\gamma}_{t_j-s} K_\gamma(x)\right| ds \quad (4.105)$$

Using the bounds provided in Proposition B.1.6, we obtain that, for any $\kappa \in (0, 1/8)$, (4.105) is bounded by

$$C\epsilon^{-1}\delta^{-3}\gamma|t-t_j|^{\frac{1}{8}-\kappa} \le C\epsilon^{-1}\delta^{-3}\gamma T J^{-\frac{1}{8}+\kappa} .$$

To complete the proof it is sufficient to choose $J = \gamma^{\lambda'}$ for λ' large enough. □

We will finally make some final remark about the expected size of the errors in Propositions 4.6.19 and 4.6.24. The key formulation of the inequalities allows us to take advantage of the separation of scales and the possibility of exploiting the fact that the process X_γ is (approximatively) the solution of the SPDE (4.8). We therefore expect $\nabla_\epsilon X_\gamma$ to posses certain behaviour.

Heuristic estimate on the size of the errors

We will now return to the one dimensional case: in the next calculation we are going to propose, under the only assumption on the initial condition given in Assumption 4.2.5, an heuristic according to which the right-hand-side of (4.104) is vanishing if $l \le \gamma^{-\vartheta}$ for any $\vartheta < 5/9$. In order to complete the replacement it is necessary to provide a bound for $\left\| \nabla_\epsilon \tilde{X}_\gamma(s)\right\|_{L^\infty(\Lambda_\varepsilon)}$. At the state of the art we do not have a satisfying bound, but we are now going to propose some argument according to which

$$\limsup_{\gamma\to 0} \gamma^{\frac{1}{3}} \mathbb{E}\left[\sup_{0\le s\le t} s^{\frac{1}{4}} \left\| \nabla_\epsilon \tilde{X}_\gamma(s)\right\|^2_{L^\infty(\Lambda_\varepsilon)}\right] < \infty \quad (4.106)$$

and from this bound we are going to conclude that for $\vartheta < 5/9$ and any $l \le \gamma^{-\vartheta}$, the right-hand-side of (4.104) is going to 0 as a small power of γ.

Let us assume for a moment (4.106).

Recall that, uniformly on π, we have the bound on the entropy $H(\pi/\mu_\gamma) \le C\epsilon^{-1}$ and optimizing in Γ the right-hand-side of (4.104) and using the scaling (4.20), we obtain that

$$\begin{aligned}
&\mathbb{E}_\pi \left[\sup_{z\in\Lambda_\varepsilon} \sup_{0\le t\le T} \left| \int_0^t \epsilon \sum_{x\in\Lambda_\varepsilon} \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma(x-z) \right.\right. \\
&\qquad \left.\left. \times\, \delta^{-2} \bigg(\sigma_x(\alpha^{-1}s)\sigma_{x+1}(\alpha^{-1}s) - \Psi^l_x(\sigma(\alpha^{-1}s)) \bigg) \nabla_\epsilon \tilde{X}_\gamma(x,s)\, ds \right| \right] \\
&\qquad \lesssim \gamma^{\frac{\lambda}{2}} + \gamma l^{\frac{3}{2}} \mathbb{E}_\pi \left[\int_0^t \left\| \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma \right\|^2_{L^2(\Lambda_\varepsilon)} \left\| \nabla_\epsilon \tilde{X}_\gamma(s) \right\|^2_{L^\infty(\Lambda_\varepsilon)} ds \right]^{\frac{1}{2}} \\
&\qquad \lesssim \gamma^{\frac{\lambda}{2}} + \gamma l^{\frac{3}{2}} \log^{\frac{1}{2}}(\gamma^{-1}) \mathbb{E}_\pi \left[\int_0^t s^{-\frac{1}{4}} (t-s)^{-\frac{3}{4}} s^{\frac{1}{4}} \left\| \nabla_\epsilon \tilde{X}_\gamma(s) \right\|^2_{L^\infty(\Lambda_\varepsilon)} ds \right]^{\frac{1}{2}} \\
&\qquad \lesssim \gamma^{\frac{\lambda}{2}} + \gamma^{\frac{5}{6}} \log^{\frac{1}{2}} l^{\frac{3}{2}}
\end{aligned}$$

Where we used (4.106) and the bound

$$\left\| \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma \right\|^2_{L^2(\Lambda_\varepsilon)} \lesssim \sum_{\omega\in\Lambda_N} |\omega \hat{K}_\gamma(\omega)|^2 |\hat{P}^{K,\gamma}_{t-s}(\omega)|^2 \lesssim |t-s|^{-\frac{3}{4}} \log(\gamma^{-1})$$

We are now going to provide some intuition behind (4.106). Here we will use without mentioning the bounds about the Fourier transform $\hat{K}_\gamma, \hat{P}^{K,\gamma}_{t-s}$ proven in Propositions B.0.1 and B.1.3.

At first approximation, the process X_γ can be decomposed into the sum

$$\nabla_\epsilon X_\gamma(x,s) = \nabla_\epsilon P^{K,\gamma}_s X^0_\gamma(x) + \nabla_\epsilon Z_\gamma(x,s) + \dots$$

where the higher order terms are supposed to be more regular. From the same considerations in the proof of Proposition 4.4.1 we have that

$$\begin{aligned}
\mathbb{E}_\pi \left[\sup_{0\le s\le t} \|\nabla_\epsilon Z_\gamma(s)\|^2_{L^\infty(\Lambda_\varepsilon)} \right] &\lesssim \sum_{\omega\in\Lambda_N} |\omega|^4 |\hat{K}_\gamma(\omega)|^2 \int_0^t |\hat{P}^{K,\gamma}_{t-s}(\omega)|^2 ds \\
&\lesssim \|K_\gamma\|^2_{\Lambda_\varepsilon} \lesssim \gamma^{-\frac{1}{3}} \ .
\end{aligned}$$

It remains to provide a bound for the term depending on the initial condition. Using Assumption 4.2.5 for $\lambda = \frac{1}{3}$,

$$\begin{aligned}
&\sup_{0\le s\le T} s^{\frac{1}{4}} \left\| \nabla_\epsilon P^{K,\gamma}_s X^0_\gamma \right\|^2_{L^\infty(\Lambda_\varepsilon)} \\
&\qquad \le \sup_{0\le s\le T} s^{\frac{1}{4}} \left\| P^{K,\gamma}_s \right\|^2_{H^{\frac{1}{2}}(\Lambda_\varepsilon)} \left\| X^0_\gamma \right\|^2_{H^{\frac{1}{2}}(\Lambda_\varepsilon)} \le C(T) \gamma^{-\frac{1}{3}} \left\| X^0_\gamma \right\|^2_{H^{\frac{1}{2}}(\Lambda_\varepsilon)}
\end{aligned}$$

where in the last line we used

$$\left\|P_s^{K,\gamma}\right\|^2_{H^{\frac{1}{2}}(\Lambda_\varepsilon)} \lesssim \sum_{\omega\in\Lambda_N} |\omega||\hat{P}_s^{K,\gamma}(\omega)|^2 \lesssim \gamma^{-\frac{1}{3}}s^{-\frac{1}{4}}+1\ .$$

By the Assumption 4.2.5 the $H^{\frac{1}{2}}(\Lambda_\varepsilon)$ of X^0_γ norm is bounded as γ vanishes. Therefore

$$\begin{aligned}
&\gamma^{\frac{1}{3}}\mathbb{E}_\pi\left[\sup_{0\le s\le T} s^{\frac{1}{4}}\left\|\nabla_\epsilon\,\tilde{X}_\gamma(s)\right\|^2_{L^\infty(\Lambda_\varepsilon)}\right]\\
&\quad\lesssim \gamma^{\frac{1}{3}}\sup_{0\le s\le T} s^{\frac{1}{4}}\left\|\nabla_\epsilon\, P_s^{K,\gamma}X^0_\gamma\right\|^2_{L^\infty(\Lambda_\varepsilon)} + \gamma^{\frac{1}{3}}\mathbb{E}_\pi\left[\sup_{0\le s\le T} s^{\frac{1}{4}}\left\|\nabla_\epsilon\, Z_\gamma(s)\right\|^2_{L^\infty(\Lambda_\varepsilon)}\right]\\
&\qquad\qquad\lesssim \left(\left\|X^0_\gamma\right\|^2_{H^{\frac{1}{2}}(\Lambda_\varepsilon)}+1\right)\ .
\end{aligned}$$

This is only an heuristic calculation since we neglected the terms coming from the nonlinear part of the discrete dynamic. We conjecture however that this heuristic can be made rigorous using the replacement lemmas of this sections for the choice $\phi(x,s)=\nabla_\epsilon\, P^{K,\gamma}_{t-s}(x)$.

The two block estimate

In Propositions 4.6.12 and 4.6.19 we effectively replaced the local function $\sigma_x\sigma_{x+1}$ with $\Psi^l_x(\sigma)$ defined in (4.84). In the equilibrium case, we were able to push the size of the block l up to macroscopic scale γ^{-1}, while in general, in dimension 1 under Assumption 4.2.5, it is possible to choose l essentially up to $\gamma^{-\frac{5}{9}}$. The aim of this subsection is to provide some arguments towards the replacement $\Psi^l_x(\sigma)$ with

$$(1+\mathfrak{g}_\gamma)h_\gamma(x)h_\gamma(x+1)-\mathfrak{g}_\gamma = (1+\mathfrak{g}_\gamma)\sum_{i\neq j}\kappa_\gamma(x-i)\kappa_\gamma(x+1-j)\sigma_i\sigma_j\ .$$

In order to do so it is convenient to have an explicit description of $\Psi^l_x(\sigma)$ in terms of the internal magnetization $\bar{\sigma}^l_x$. Lemma 4.6.17 quantifies the difference between $\Psi^l_x(\sigma)$ and the more convenient function of the magnetization

$$-\frac{1}{(2d+1)^d-1}+\frac{(2d+1)^d}{(2d+1)^d-1}\left(\bar{\sigma}^l_x\right)^2 = \mathrm{Av}_{i\neq j\in B^l_x}\,\sigma_i\sigma_j\ .$$

We resume this considerations into the following proposition, whose proof is a direct application of Lemma 4.6.17.

Proposition 4.6.25 *For $l \leq \mathbf{a}\gamma^{-1}$, there exists $C > 0$ such that*

$$\begin{aligned}
&\mathbb{E}_\pi \Bigg[\sup_{z\in\Lambda_\varepsilon} \sup_{0\le t\le T} \Bigg| \int_0^t \epsilon^d \sum_{x\in\Lambda_\varepsilon} \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma(x-z) \, \nabla_\epsilon \tilde{X}_\gamma(x,s) \\
&\quad \times \delta^{-2} \left\{ \Psi^l_x(\sigma) + \frac{1}{(2d+1)^d - 1} - \frac{1}{1-(2d+1)^{-d}} \left(\bar{\sigma}^l_x\right)^2 \right\} ds \Bigg| \Bigg] \\
&\quad \le C\mathbb{E}_\pi \left[\sup_{0\le t\le T} \int_0^t \left\| \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma \right\|_{L^1(\Lambda_\varepsilon)} \epsilon\delta^{-1} l \left\| \nabla_\epsilon X_\gamma(s)\right\|^2_{L^\infty(\Lambda_\varepsilon)} \, ds \right] \\
&\quad + C\mathbb{E}_\pi \left[\sup_{0\le t\le T} \int_0^t \left\| \nabla_\epsilon P^{K,\gamma}_{t-s} K_\gamma \right\|_{L^1(\Lambda_\varepsilon)} \delta^{-2} l^{-2d} \left\| \nabla_\epsilon X_\gamma(s)\right\|_{L^\infty(\Lambda_\varepsilon)} \, ds \right] \qquad (4.107)
\end{aligned}$$

A more general way to obtain similar approximations is via an equivalence of ensembles, as in [GJS15], however we found the approach in Lemma 4.6.17 more direct and convenient. We would like to remark that, in dimension 1, using the heuristic (4.106), the right-hand-side of (4.107) is of order

$$l\gamma^{\frac{2}{3}} + l^{-2}\gamma^{-\frac{5}{6}}$$

which is negligible as long as $\gamma^{-\frac{5}{12}} \ll l \ll \gamma^{-\frac{5}{9}}$.

Having a more convenient description for $\Psi^l_x(\sigma)$, we are now going to estimate the cost of the replacement

$$\mathrm{Av}_{i\neq j\in B^l_x} \sigma_i\sigma_j \sim (1+\mathfrak{g}_\gamma) h_\gamma(x) h_\gamma(x+1) - \mathfrak{g}_\gamma \, .$$

Let us introduce

$$\begin{aligned}
F_{\epsilon x, l}(\sigma) &= (1+\mathfrak{g}_\gamma)^{-1} \Big[\mathrm{Av}_{i_1\neq j_1 \in B^l_x} \sigma_{i_1}\sigma_{j_1} - h_\gamma(x)h_\gamma(x+1) + (1+\mathfrak{g}_\gamma)\mathfrak{g}_\gamma \Big] \nabla^+_N \tilde{h}_\gamma(x) \\
&= \sum_{i_2\neq j_2} \kappa_\gamma(i_2 - x)\kappa_\gamma(j_2-x-1) \, \mathrm{Av}_{i_1\neq j_1\in B^l_x} [\sigma_{i_1}\sigma_{j_1} - \sigma_{i_2}\sigma_{j_2}] \nabla^+_N \tilde{h}_\gamma(x) \\
&= \sum_{i_2\neq j_2 \neq u} \nabla^+_N \tilde{\kappa}_\gamma(x-u) \kappa_\gamma(i_2-x)\kappa_\gamma(j_2-x-1) \, \mathrm{Av}_{i_1\neq j_1\in B^l_x} \left(\sigma_{i_1}\sigma_{j_1} - \sigma_{i_2}\sigma_{j_2}\right) \sigma_u \\
&\quad + \frac{1}{2} \sum_{u\in\Lambda_N} |\nabla^+_N \tilde{\kappa}_\gamma(x-u)|^2 \nabla^+_N \tilde{h}_\gamma(x)
\end{aligned}$$

where in the second line we restricted the summation on $u \neq i_2, j_2$ at the cost of

$$\frac{1}{2} \sum_{u\in\Lambda_N} |\nabla_N \tilde{\kappa}_\gamma(x-u)|^2 \nabla^+_N \tilde{h}_\gamma(x) \sim \mathcal{O}(\gamma^3) \nabla^+_N \tilde{h}_\gamma(x)$$

which is going to be negligible in the limit (here we used the fact that $|\nabla_N \tilde{\kappa}_\gamma(x-u)| \leq C\gamma^2$). In the following calculations, to keep the notations light, we will use the definition

$$h_\gamma^{\{v,r\}}(u) = \sum_{i \in \Lambda_N \setminus \{v,r\}} \kappa_\gamma(u-i)\sigma_i \ .$$

Consider now

$$(\sigma_{i_1}\sigma_{j_1} - \sigma_{i_2}\sigma_{j_2}) = \sigma_{i_1}\left(\sigma_{j_1} - \sigma_{j_2}\right) + \left(\sigma_{i_1} - \sigma_{i_2}\right)\sigma_{j_2}$$

and use the fact that

$$\bar{\sigma}_x^l = \mathrm{Av}_{j_1 \in B_x^l}\, \mathrm{Av}_{i_1 \in B_x^l \setminus \{j_1+1\}}\, \sigma_{i_1}$$

to decompose $F_{\epsilon x,l}(\sigma) = F^{(1)}_{\epsilon x,l}(\sigma) + F^{(2)}_{\epsilon x,l}(\sigma) + \mathcal{O}(\gamma^3)$ where

$$\begin{aligned}
F^{(1)}_{\epsilon x,l}(\sigma) &= \sum_{\substack{j_2 \in \Lambda_N \\ j_1 \in B_x^l}} \frac{\kappa_\gamma(j_2 - x)}{(2l+1)^d}\left(\sigma_{j_1} - \sigma_{j_2}\right)\left(\mathrm{Av}_{i_1 \in B_x^l \setminus \{j_1\}}\, \sigma_{i_1}\right)\nabla_N^+ \tilde{h}_\gamma^{\{j_2\}}(x) \\
F^{(2)}_{\epsilon x,l}(\sigma) &= \sum_{i_2 \in \Lambda_N} \kappa_\gamma(i_2 - x)\, \mathrm{Av}_{i_1 \in B_x^l}\left(\sigma_{i_1} - \sigma_{i_2}\right)\left(\tilde{h}_\gamma^{\{i_1,i_2\}}(x+1)\nabla_N^+ h_\gamma^{\{i_1,i_2\}}(x)\right) \ .
\end{aligned}$$

We would like to replace, in the above equation, the difference $\sigma_{j_1} - \sigma_{j_2}$ with a similar function having zero mean according to any canonical measure $\mu_\gamma^{\Lambda,\eta,M}\left[\,\cdot\,\right]$ for any $\Lambda \subseteq \Lambda_N$ containing the sites $\{i_1, i_2\}$. Recall the definition of the current $w_{x,y}$ in (4.5) and the fact that, using the detailed balance conditions (DB), $w_{x,y}$ satisfies

$$w_{x,y}(\sigma^{x,y})\mu_{\gamma,b}(\sigma^{x,y}) = -w_{x,y}(\sigma)\mu_{\gamma,b}(\sigma) \tag{4.108}$$

and this guarantees that $w_{x,y}$ has zero mean with respect to any canonical Gibbs measure restricted to any block containing the set $\{x,y\}$. Moreover, from the definition (4.3)

$$w_{x,y}(\sigma) = (\sigma_x - \sigma_y) - (1 - \sigma_x\sigma_y)\tanh\left(\beta[\tilde{h}_\gamma(y,t) - \tilde{h}_\gamma(x,t)]\right) \tag{4.109}$$

we can decompose

$$F^{(1)}_{\epsilon x,l}(\sigma) + F^{(2)}_{\epsilon x,l}(\sigma) = \tilde{F}^{(1)}_{\epsilon x,l}(\sigma) + \tilde{F}^{(2)}_{\epsilon x,l}(\sigma) + \tilde{G}^{(1)}_{\epsilon x,l}(\sigma) + \tilde{G}^{(2)}_{\epsilon x,l}(\sigma)$$

where the main terms are

$$\tilde{F}^{(1)}_{\epsilon x,l}(\sigma) = \sum_{\substack{j_2\in\Lambda_N\\ j_1\in B^l_x}} \frac{\kappa_\gamma(j_2-x)}{(2l+1)^d} w_{j_1,j_2}(\sigma)\left(\mathrm{Av}_{i_1\in B^l_x\setminus\{j_1\}}\,\sigma_{i_1}\right)\nabla^+_N \tilde{h}^{\{j_2\}}_\gamma(x)$$

$$\tilde{F}^{(2)}_{\epsilon x,l}(\sigma) = \sum_{i_2\in\Lambda_N} \kappa_\gamma(i_2-x)\,\mathrm{Av}_{i_1\in B^l_x}\, w_{i_1,i_2}(\sigma)\left(h^{\{i_1,i_2\}}_\gamma(x+1)\nabla^+_N \tilde{h}^{\{i_1,i_2\}}_\gamma(x)\right)$$

and the rest of the terms are

$$\tilde{G}^{(1)}_{\epsilon x,l}(\sigma) = \sum_{\substack{j_2\in\Lambda_N\\ j_1\in B^l_x}} \frac{\kappa_\gamma(j_2-x)}{(2l+1)^d}(1-\sigma_{j_1}\sigma_{j_2})\tanh\left(\beta[\tilde{h}_\gamma(j_1,t)-\tilde{h}_\gamma(j_2,t)]\right)$$
$$\times\left(\mathrm{Av}_{i_1\in B^l_x\setminus\{j_1\}}\,\sigma_{i_1}\right)\left(\nabla^+_N \tilde{h}^{\{j_2\}}_\gamma(x)\right) \tag{4.110}$$

$$\tilde{G}^{(2)}_{\epsilon x,l}(\sigma) = \sum_{\substack{i_2\in\Lambda_N\\ i_1\in B^l_x}} \frac{\kappa_\gamma(i_2-x)}{(2l+1)^d}(1-\sigma_{i_1}\sigma_{i_2})\tanh\left(\beta[\tilde{h}_\gamma(i_1,t)-\tilde{h}_\gamma(i_2,t)]\right)$$
$$\times\left(h^{\{i_1,i_2\}}_\gamma(x+1)\nabla^+_N \tilde{h}^{\{i_1,i_2\}}_\gamma(x)\right) \tag{4.111}$$

The next lemma provides a bound over the time integral of $\tilde{F}^{(1)}_{\epsilon x,l}$ and $\tilde{F}^{(2)}_{\epsilon x,l}$, since these are the main components when replacing averages over blocks of size l with blocks of order γ^{-1}, we will refer to the next lemma as the two blocks estimate.

Lemma 4.6.26 (Two-blocks estimate) *Let $0 < l \le \mathbf{a}\gamma^{-1}$, there exists $C > 0$ such that for all $\Gamma > 0$*

$$\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t \sum_{x\in\Lambda_\varepsilon} \epsilon^d\,\nabla_\epsilon\,\phi(x-z,s)\delta^{-3}\epsilon^{-1}\tilde{F}^{(1)}_{x,l}(\sigma(\alpha^{-1}s))\,ds\right|\right]$$
$$\le \Gamma^{-1}\left(H(\pi/\mu_\gamma)+2d\log(\epsilon^{-1})\right)$$
$$+\,C\Gamma\gamma^{-2}\alpha\delta^{-4}\epsilon^d\mathbb{E}_\pi\int_0^t \sup_{z\in\Lambda_\varepsilon}\sum_{x\in\Lambda_\varepsilon}\epsilon^d|\nabla_\epsilon\,\phi(x-z,s)|^2$$
$$\times\left(|\bar{\sigma}^l_{\epsilon^{-1}x}(\alpha^{-1}s)|+l^{-d}\right)^2\left(\left|\nabla_\epsilon\,\tilde{X}_\gamma(x,s)\right|+1\right)^2\,ds \tag{4.112}$$

$$\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t\sum_{x\in\Lambda_\varepsilon}\epsilon^d\,\nabla_\epsilon\,\phi(x-z,s)\delta^{-3}\epsilon^{-1}\tilde{F}^{(2)}_{x,l}(\sigma(\alpha^{-1}s))\,ds\right|\right]$$
$$\leq\Gamma^{-1}\left(H(\pi/\mu_\gamma)+2d\log(\epsilon^{-1})\right)$$
$$+C\Gamma\gamma^{-2}\alpha\delta^{-2}\epsilon^d\mathbb{E}_\pi\int_0^t\sup_{z\in\Lambda_\varepsilon}\sum_{x\in\Lambda_\varepsilon}\epsilon^d|\,\nabla_\epsilon\,\phi(x-z,s)|^2$$
$$\times\left(|X_\gamma(x,s)|+\gamma\delta^{-1}\right)^2\left(\left|\,\nabla_\epsilon\,\tilde{X}_\gamma(x,s)\right|+1\right)^2\,ds \tag{4.113}$$

Remark 4.6.27 It is easy to see that the same proof of Proposition 4.6.24 allows to take a supremum both in space and time in (4.112) and (4.113), at an additive cost of γ^λ, for all $\lambda>0$ provided we allow the constant C to depend on λ.
Furthermore, from the heuristic in (4.106), it is possible to see that, in 1 dimension, for a general initial measure π with $H(\pi/\mu_\gamma)\lesssim\epsilon^{-1}$ the left-hand-side of (4.113) is bounded by

$$\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon,0\leq t\leq T}\left|\int_0^t\sum_{x\in\Lambda_\varepsilon}\epsilon^d\,\nabla_\epsilon\,P^{K,\gamma}_{t-s}K_\gamma(x-z)\delta^{-3}\epsilon^{-1}\tilde{F}^{(2)}_{x,l}(\sigma(\alpha^{-1}s))\,ds\right|\right]$$
$$\lesssim\gamma^{\lambda/2}+\left\{\gamma^{\frac{2}{3}}\int_0^T\left\|\nabla_\epsilon\,P^{K,\gamma}_{T-s}K_\gamma\right\|^2_{L^2(\Lambda_\varepsilon)}\mathbb{E}_\pi\left[\|X_\gamma(s)\,\nabla_\epsilon\,X_\gamma(s)\|^2_{L^\infty(\Lambda_\varepsilon)}\right]\right\}^{\frac{1}{2}}$$
$$\lesssim\gamma^{\lambda/2}+\gamma^{\frac{1}{6}}\;.$$

We also remark that (4.112) depends on $\bar{\sigma}^l_x$, whose difference with $h_\gamma(x)$ can potentially be estimated using a second time the two block estimate. Alternatively (4.112) is sufficient when $\pi=\mu_\gamma$.

Proof. We are proving the result for $\tilde{F}^{(1)}$, the same proof can be applied to $\tilde{F}^{(2)}$ as well. The proof will follow the lines of Proposition 4.6.12. Define, for $z\in\Lambda_\varepsilon$

$$\psi_1(z,s)=\sum_{x\in\Lambda_\varepsilon}\epsilon^d\,\nabla_\epsilon\,\phi(x-z,s)\delta^{-3}\epsilon^{-1}\tilde{F}^{(1)}_{x,l}(\sigma(\alpha^{-1}s))$$
$$\psi_2(z,s)=\sum_{x\in\Lambda_\varepsilon}\epsilon^d|\,\nabla_\epsilon\,\phi(x-z,s)|^2\left(|\bar{\sigma}^l_{\epsilon^{-1}x}(\alpha^{-1}s)|+l^{-d}\right)^2\left(\left|\,\nabla_\epsilon\,\tilde{X}_\gamma(x,s)\right|+1\right)^2\;.$$

Write, for $\theta>0$ to be chosen later

$$\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t\psi_1(z,s)ds\right|\right]=$$
$$\theta\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\int_0^t\psi_2(z,s)ds\right]+\mathbb{E}_\pi\left[\sup_{z\in\Lambda_\varepsilon}\left|\int_0^t\psi_1(z,s)ds\right|-\theta\int_0^t\psi_2(z,s)ds\right]$$

Using the entropy inequality, for $\Gamma > 0$, bound the last term with

$$\Gamma^{-1} H(\pi/\mu_\gamma) + \Gamma^{-1} \log \mathbb{E}_{\mu_\gamma} \left[\sup_{z \in \Lambda_\varepsilon} \exp \left\{ \Gamma \left| \int_0^t \psi_1(z,s)\, ds \right| - \Gamma\theta \int_0^t \psi_2(z,s)\, ds \right\} \right] .$$

We bound the supremum with the summation over $z \in \Lambda_\varepsilon$. Lemma 4.6.28 shows that, for $\theta = C\Gamma\gamma^{-2}\alpha\delta^{-4}\epsilon^d$

$$\Gamma^{-1} \log \sum_{z \in \Lambda_\varepsilon} \mathbb{E}_{\mu_\gamma} \left[\exp \left\{ \Gamma \left| \int_0^t \psi_1(z,s)\, ds \right| - \Gamma\theta \int_0^t \psi_2(z,s)\, ds \right\} \right] \leq \Gamma^{-1} \log(\epsilon^{-2d})$$

and putting the estimates together we obtain

$$\begin{aligned} &\mathbb{E}_\pi \left[\sup_{z \in \Lambda_\varepsilon} \left| \int_0^t \psi_1(z,s) ds \right| \right] \\ &\qquad \leq \Gamma^{-1} \left(H(\pi/\mu_\gamma) + 2d \log(\epsilon^{-1}) \right) + \Gamma\gamma^{-2}\alpha\delta^{-4}\epsilon^d \mathbb{E}_\pi \left[\sup_{z \in \Lambda_\varepsilon} \int_0^t \psi_2(z,s) ds \right] \end{aligned}$$

□

Lemma 4.6.28 *Let ψ_1, ψ_2 be defined as in the proof of Lemma 4.6.26. Then for $0 < l \leq \mathbf{a}\gamma^{-1}$, then, there exists $C > 0$ such that for any $\Gamma > 0$*

$$\log \mathbb{E}_{\mu_\gamma} \left[\exp \left\{ \Gamma \left| \int_0^t \psi_1(z,s)\, ds \right| - C\Gamma^2\gamma^{-2}\alpha\delta^{-4}\epsilon^d \int_0^t \psi_2(z,s)\, ds \right\} \right] \leq 0$$

Proof. Using the fact that $\exp\{|\Gamma\psi_1(z,s)|\} \leq \exp\{\Gamma\psi_1(z,s)\} + \exp\{-\Gamma\psi_1(z,s)\}$ and that μ_γ is translation invariant, it is sufficient to provide a bound of the form

$$\log \mathbb{E}_{\mu_\gamma} \left[\exp \left\{ \pm\Gamma \int_0^t \psi_1(0,s)\, ds - \theta \int_0^t \psi_2(0,s)\, ds \right\} \right] \leq 0$$

where θ will be chosen later. As in the proof of Lemma 4.6.22, we use the Feynman-Kac formula [KL99, Appendiix 1, Lemma 7.2]

$$\begin{aligned} &\log \mathbb{E}_{\mu_\gamma} \exp \left(\left\{ \int_0^t \pm\Gamma\psi_1(0,s)\, ds - \theta\psi_2(0,s) ds \right\} \right) \\ &\qquad \leq \int_0^t \sup \operatorname{spec}_{L^2(\mu_\gamma)} \left\{ \pm\Gamma\psi_1(0,s) - \theta\psi_2(0,s) + \alpha^{-1}\mathscr{L}^K \right\} ds \end{aligned}$$

where the quantity inside the integral denotes the largest eigenvalue of the self-adjoint operator. By the variational formula (4.97), the largest eigenvalue is bounded by

$$\sup_{g:\mu_\gamma[g^2]=1} \mu_\gamma \left[\sum_{x\in\Lambda_\varepsilon} \pm\Gamma\, \nabla_\epsilon\, \phi(x,s)\delta^{-3}\epsilon^1 \tilde{F}^{(1)}_{x,l}(\sigma(\alpha^{-1}s))g^2(\sigma) - \theta\psi_2(x,s)g^2(\sigma) \right] \\ - \alpha^{-1}\mathbf{D}_\gamma^{\Lambda_N}(g)\,. \quad (4.114)$$

To prove (4.112) it is sufficient to show that (4.114) is negative for θ of order γ^2. Using the definition of $\tilde{F}^{(1)}_{x,l}(\sigma)$ we obtain

$$\mu_\gamma\left[\tilde{F}^{(1)}_{x,l}(\sigma)g^2(\sigma)\right] = \sum_{j_2\in\Lambda_N} \mathrm{Av}_{j_1\in B^l_x}\, \kappa_\gamma(j_2-x) \\ \times \mu_\gamma\left[\mu_\gamma\left[w_{j_1,j_2}(\sigma)g^2(\sigma)\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right]\left(\mathrm{Av}_{i_1\in B^l_x\setminus\{j_1\}}\,\sigma_{i_1}\right)\nabla^+_N\tilde{h}^{\{j_2\}}_\gamma(x)\right]$$

where we conditioned over $\mathcal{F}_{\{j_1,j_2\}^c}$ and we used the fact that $w_{j_1,j_2}g$ are the only quantity that can depend on the spins in $\{j_1,j_2\}$. We now take advantage of (4.108) that implies

$$\mu_\gamma\left[w_{j_1,j_2}(\sigma)g^2(\sigma)\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right] = -\mu_\gamma\left[w_{j_1,j_2}(\sigma)g^2(\sigma^{\{j_1,j_2\}})\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right]$$

and therefore we have, using Young's inequality, that for any $\mathcal{F}_{\{j_1,j_2\}^c}$- measurable function $Y(\sigma)$ and for positive $\lambda>0$

$$\left|\mu_\gamma\left[Y(\sigma)w_{j_1,j_2}(\sigma)g^2(\sigma)\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right]\right| \\ = \frac{1}{2}\left|Y(\sigma)\mu_\gamma\left[w_{j_1,j_2}(\sigma)\left(g^2(\sigma^{\{j_1,j_2\}})-g^2(\sigma)\right)\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right]\right| \\ \lesssim \mu_\gamma\left[\lambda Y^2(\sigma)\left(g(\sigma^{\{j_1,j_2\}})+g(\sigma)\right)^2+\lambda^{-1}\left(g(\sigma^{\{j_1,j_2\}})-g(\sigma)\right)^2\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right].$$

Moreover, by (DB) and (CB)

$$\mu_\gamma\left[g^2(\sigma^{\{j_1,j_2\}})\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right] \lesssim \mu_\gamma\left[g^2(\sigma)\middle|\mathcal{F}_{\{j_1,j_2\}^c}\right]$$

Choosing $Y(\sigma) = \Gamma\,\nabla_\epsilon\,\phi(\epsilon x,s)\delta^{-3}\epsilon^{d-1}\left(\mathrm{Av}_{i_1\in B^l_x\setminus\{j_1\}}\,\sigma_{i_1}\right)\nabla^+_N\tilde{h}^{\{j_2\}}_\gamma(x)$ we can bound

(4.114) with

$$\sum_{x,j_2\in\Lambda_N} \mathrm{Av}_{j_1\in B_x^l}\,\kappa_\gamma(j_2-x)\Bigg\{\lambda^{-1}\mu_\gamma\Big[\Big(g(\sigma^{\{j_1,j_2\}})-g(\sigma)\Big)^2\Big]$$

$$+\lambda\Gamma^2\epsilon^{2d-2}\delta^{-6}|\,\nabla_\epsilon\,\phi(\epsilon x,s)|^2\mu_\gamma\Big[\Big(|\bar\sigma_x^l|+l^{-d}\Big)^2\left|\nabla_N^+\tilde h_\gamma^{\{j_2\}}(x)\right|^2 g^2\Big]\Bigg\}$$

$$-\,\theta\mu_\gamma\left[\psi_2(0,s)g^2(\sigma)\right]-\alpha^{-1}\mathbf{D}_\gamma^{\Lambda_N}(g)$$

By the moving particle lemma (Lemma 4.6.10) and the fact that the support of κ_γ has diameter γ^{-1} we have

$$\sum_{x,j_2\in\Lambda_N} \mathrm{Av}_{j_1\in B_x^l}\,\kappa_\gamma(j_2-x)\mu_\gamma\Big[\Big(g(\sigma^{\{j_1,j_2\}})-g(\sigma)\Big)^2\Big]\le C\gamma^{-2}\mathbf{D}_\gamma^{\Lambda_N}(g)$$

choosing $\lambda=C\gamma^{-2}\alpha$ and $\theta\sim\Gamma^2\gamma^{-2}\alpha\delta^{-4}\epsilon^d$ concludes the proof. □

In order to complete the substitution one needs to provide a bound for $\tilde G^{(1)}_{\epsilon x,l}(\sigma)$ and $\tilde G^{(2)}_{\epsilon x,l}(\sigma)$, defined respectively in (4.110), (4.111). At the state of the art we do not have a sufficient control over $\tilde G^{(1)}_{\epsilon x,l}(\sigma)$, we just would like to highlight that the form of (4.110), (4.111) allows a possible application of a second 2-blocks estimate, hence a possible replacement of $\sigma_{j_1}\sigma_{j2}$ with $(1+\mathfrak{g}_\gamma)h_\gamma(j_1)h_\gamma(j_2)-\mathfrak{g}_\gamma$. This procedure might be successful considering the fact that (4.110) and (4.111) have extra powers of the magnetization field.

Appendix A

Discrete Besov spaces

We will collect here some of the definitions and proprieties of Besov spaces used in the previous chapters. The definitions and proofs in this appendix are based upon [MW17a, MW17b] and [BCD11]. In [BCD11, Prop. 2.10] it is proven the existence of continuous functions $\tilde{\chi}, \chi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned} &\mathrm{supp}(\tilde{\chi}) \subseteq B_0(4/3) \\ &\mathrm{supp}(\chi) \subseteq B_0(8/3) \setminus B_0(3/4) \end{aligned}$$

such that $\forall r \in \mathbb{R}^d$

$$\tilde{\chi}(r) + \sum_{k=0}^{\infty} \chi(2^{-k}\, r) = 1\ .$$

Having fixed the functions $\tilde{\chi}$ and χ, define

$$\chi_{-1} \stackrel{\text{def}}{=} \tilde{\chi}, \qquad \chi_k(\cdot) \stackrel{\text{def}}{=} \chi(2^{-k}\ \cdot) \qquad \text{for } (k \geq 0)$$

For $g : \mathbb{T}^d \to \mathbb{R}$ define the projection on the k-th Paley-Littlewood block as

$$\delta_k g(x) = 2^{-d} \sum_{\omega \in \mathbb{Z}^d} \chi_k(\omega)\, \widehat{g}(\omega) e_\omega(x) \tag{A.1}$$

for $x \in \mathbb{T}^d$ and $k \geq -1$. The Paley-Littlewood projection can be seen also as a convolution operator. For this purpose define, for $k \geq -1$ and $x \in \mathbb{T}^d$

$$\tilde{\eta}_k(x) \stackrel{\text{def}}{=} 2^{-d} \sum_{\omega \in \mathbb{Z}^d} \chi_k(\omega) e_\omega(x)\ , \qquad \eta_k(x) \stackrel{\text{def}}{=} 2^{-d} \sum_{\omega \in \Lambda_N^d} \chi_k(\omega) e_\omega(x) \tag{A.2}$$

where we are abusing the notation omitting in η_k the dependency on N.
It is clear from (A.1), that $\delta_k g = \tilde{\eta}_k \star g$ on $\mathbb{T}^d$.
The continuous Besov norm is defined, for $\nu \in \mathbb{R}$, $p, q \in [1, \infty]$ and smooth functions $g : \mathbb{T}^d \to \mathbb{R}$

$$\|g\|_{\mathcal{B}^{\nu}_{p,q}} \stackrel{\text{def}}{=} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k g\|^q_{L^p(\mathbb{T}^d)} \right)^{\frac{1}{q}} \tag{A.3}$$

with the usual convention if $q = \infty$.
Recall the definition of the extension operator (1.9). We defined two possible generalization of the Besov norm for functions defined in the discrete lattice (see Section 1.2.2). The first one is obtained extending the discrete function to the whole d-dimensional torus and it is denoted with $\|\cdot\|_{\mathcal{B}^{\nu}_{p,q}}$

$$\|g\|_{\mathcal{B}^{\nu}_{p,q}} \stackrel{\text{def}}{=} \|\mathsf{Ext}(g)\|_{\mathcal{B}^{\nu}_{p,q}}$$

while the other is obtained not only extending the function with the extension operator but also performing the L^p norm in (A.3) on the discrete space Λ^d_ε instead of $\mathbb{T}^d$ and it is denoted, for $\nu \in \mathbb{R}$, $p, q \in [1, \infty]$, with $\|\cdot\|_{\mathcal{B}^{\nu}_{p,q}(\Lambda^d_\varepsilon)}$

$$\|g\|_{\mathcal{B}^{\nu}_{p,q}(\Lambda^d_\varepsilon)} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{k \geq -1} 2^{\nu k q} \|\delta_k \mathsf{Ext}(g)\|^q_{L^p(\Lambda^d_\varepsilon)} \right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{k \geq -1} 2^{\nu k} \|\delta_k \mathsf{Ext}(g)\|_{L^p(\Lambda^d_\varepsilon)} & \text{if } q = \infty \end{cases} \tag{A.4}$$

It is clear, from the definitions of $\mathsf{Ext}(g)$, (A.1) and (A.2), that for $x \in \Lambda^d_\varepsilon$

$$\delta_k \mathsf{Ext}(g)(x) = 2^{-d} \sum_{\omega \in \Lambda^d_N} \chi_k(\omega)\, \widehat{g}(\omega) e_\omega(x) = \eta_k *_\epsilon f(x) \qquad \text{for } x \in \Lambda_\varepsilon$$

where $\eta_k(x)$ is defined in (A.2).

The next lemma is a minor generalisation of [MW17a, Lemma B.6].

Lemma A.0.1 *For $p \in [1, \infty]$ and $\kappa > 0$, there exists a constant C such that for all $f : \Lambda_\varepsilon \to \mathbb{R}$,*

$$\|\mathsf{Ext}(f)\|_{L^p(\mathbb{T}^2)} \leq C \log^2(\epsilon^{-1}) \|f\|_{L^p(\Lambda_\varepsilon)} \tag{A.5}$$

$$\|\mathsf{Ext}(f)\|_{L^p(\mathbb{T}^2)} \leq C \|f\|_{L^p(\Lambda_\varepsilon)} + C\epsilon^{-\kappa} \|f\|^{1-\frac{1}{p}}_{L^{2p-2}(\Lambda_\varepsilon)} \left\{ \sum_{\substack{|x-y|=\epsilon \\ x,y \in \Lambda_\varepsilon}} \epsilon^2 (f(y) - f(x))^2 \right\}^{\frac{1}{2p}} . \tag{A.6}$$

The same lemma holds true in any dimension, with a factor $C(d) \log^d(\epsilon^{-1})$ in (A.5).

Proof. We first show (A.5). Recall that from the definition of the extension operator $\mathsf{Ext} f(x) = f(x)$ for $x \in \Lambda_\varepsilon^2$, and

$$\mathsf{Ext}(f)(x) = \sum_{z\in\Lambda_\varepsilon} \epsilon^2 f(z) \prod_{i=1,2} \frac{\sin\left(\pi\epsilon^{-1}(x_j - z_j)\right)}{2\sin\left(\frac{\pi}{2}(x_j - z_j)\right)} \qquad x \in \mathbb{T}^2 .$$

Using the inequality $\sin(2\epsilon^{-1}a)/\sin(a) \lesssim \epsilon^{-1} \wedge |a|^{-1}$ we can bound

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |f(z)| \prod_{i=1,2} \epsilon^{-1} \wedge |z_i - x_i|^{-1} .$$

For $x \in \mathbb{T}^2$, denote with $[x]_\epsilon$ the closest point to x in Λ_ε. We can then rewrite the above inequality as

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |f(z + [x]_\epsilon)| \prod_{i=1,2} \epsilon^{-1} \wedge |z_i + [x]_{\epsilon,i} - x_i|^{-1} .$$

we observe now that if $|z_i| \leq \epsilon$, then $|z_i + [x]_{\epsilon,i} - x_i|^{-1} \gtrsim \epsilon^{-1}$, while if $|z_i| > \epsilon$ we have that $|z_i + [x]_{\epsilon,i} - x_i|^{-1} \lesssim |z_i|^{-1} \lesssim \epsilon^{-1}$, hence

$$|\mathsf{Ext}(f)(x)| \lesssim \sum_{z\in\Lambda_\varepsilon} \epsilon^2 |f(z + [x]_\epsilon)| \prod_{i=1,2} \epsilon^{-1} \wedge |z_i|^{-1} ,$$

and taking the $L^p(\mathbb{T}^2, dx)$ norm of the right-hand-side yields

$$\Big(\int_{\mathbb{T}^2} |f([x]_\epsilon)|^p dx\Big)^{\frac{1}{p}} \Big(1 + 2 \sum_{1\leq k\leq \epsilon^{-1}} k^{-1}\Big)^2 \lesssim \|f\|_{L^p(\Lambda_\varepsilon)} \log^2(\epsilon^{-1}) ,$$

as claimed. The inequality (A.6) is a consequence of Hölder's inequality

$$\begin{aligned}
\|\mathsf{Ext}(f)\|^p_{L^p(\mathbb{T}^2)} &\leq \|f\|^p_{L^p(\Lambda_\varepsilon)} + \sum_{x\in\Lambda_\varepsilon} \int_{|y|\leq\epsilon/2} |\mathsf{Ext} f(x+y) - f(x)|^p d^2y \\
&\leq \|f\|^p_{L^p(\Lambda_\varepsilon)} + \|\mathsf{Ext} f\|^{p-1}_{L^{2p-2}(\mathbb{T}^2)} \Bigg(\sum_{x\in\Lambda_\varepsilon} \int_{|y|\leq\epsilon/2} |\mathsf{Ext} f(x+y) - f(x)|^2 d^2y\Bigg)^{\frac{1}{2}} \\
&\leq \|f\|^p_{L^p(\Lambda_\varepsilon)} + \|\mathsf{Ext} f\|^{p-1}_{L^{2p-2}(\mathbb{T}^2)} \Bigg(\sum_{x\in\Lambda_\varepsilon} \int_{|y|\leq\epsilon/2} |\mathsf{Ext} f(x+y) - f(x)|^2 d^2y\Bigg)^{\frac{1}{2}} \\
&\lesssim \|f\|^p_{L^p(\Lambda_\varepsilon)} + \epsilon \, \|\mathsf{Ext} f\|^{p-1}_{L^{2p-2}(\mathbb{T}^2)} \|\mathsf{Ext} f\|_{\dot H^1(\mathbb{T}^2)}
\end{aligned}$$

where we denoted by $\|\mathsf{Ext} f\|_{\dot H^1(\mathbb{T}^2)}$ the homogeneous Sobolev seminorm. From the definition of the extension operator it is easy to see that

$$\|\mathsf{Ext} f\|^2_{\dot H^1(\mathbb{T}^2)} = \sum_{\omega\in\Lambda_N} |\omega|^2 |\hat f(\omega)|^2 \lesssim \sum_{\substack{|x-y|=\epsilon \\ x,y\in\Lambda_\varepsilon}} \epsilon^2 \frac{(f(y)-f(x))^2}{\epsilon^2}\ ,$$

and an application of (A.5) yields (A.6). □

Lemma A.0.2 *Let* $\chi : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function with compact support. For every* $\lambda \in (0,1)$ *and* $p \in [0,\infty]$ *we have*

$$\sup_{\lambda\in(0,1)} \lambda^{d\left(1-\frac{1}{p}\right)} \Big\| \sum_{\omega\in\Lambda_N^d} \chi(\lambda\omega) e_\omega(x) \Big\|_{L^p(\Lambda_\varepsilon^d)} < \infty$$

and in particular $\sup_{k\geq -1} \|\eta_k(x)\|_{L^1(\Lambda_\varepsilon^d)} < C$.

The above result is proven in [MW17a, Lemma A.1] for the L^p in the whole torus, the generalisation to Λ_ε^d follows from the same argument.

Proof. Using the interpolation of the L^p norm, it is sufficient to prove the estimate for $p = 1$ and $p = \infty$. The estimate for $p = \infty$ follows from the boundedness of χ and $\sum_{\omega\in\Lambda_N^d} \chi(\lambda\omega) e_\omega(x) \lesssim \lambda^d$. Assume $p = 1$ and define $\check\chi_\lambda(x) = \lambda^d \sum_{\omega\in(\lambda\Lambda_N)^d} \chi(\omega) e_\omega(x)$. Then

$$\sum_{x\in\Lambda_\varepsilon} \epsilon^d \Big| \sum_{\omega\in\Lambda_N^d} \chi(\lambda\omega) e_\omega(x) \Big| = \sum_{x\in\lambda^{-1}\Lambda_\varepsilon} \lambda^{-d} |\check\chi_\lambda(x)|\ .$$

It is then sufficient to prove for k sufficiently large that $|\check\chi_\lambda(x)| \leq C(k)|x|^{-2k}$ for $|x| \leq \lambda^{-1}$. Define the operator $\underline{\Delta}_\lambda$ differentiating with respect to the variable ω, which we think in $\mathbb{R}^d$

$$\underline{\Delta}_\lambda f(\omega) = \lambda^{-2} \sum_{i=1}^d f(\omega + \lambda e_i) - 2f(\omega) + f(\omega - \lambda e_i)$$

in particular we have that $\underline{\Delta}_\lambda e_\omega(x) = 2\lambda^{-2} \sum_{i=1}^d (\cos(\pi\lambda x_i) - 1) e_\omega(x)$ for $\omega \in (\lambda\Lambda_N)^d$ Therefore

$$\check\chi_\lambda(x) = \frac{1}{2}\lambda^2 \left(\sum_{i=1}^d \cos(\pi\lambda x_i) - 1 \right)^{-1} \sum_{\omega\in(\lambda\Lambda_N)^d} \lambda^d \chi(\omega) \underline{\Delta}_\lambda e_\omega(x)$$

and, by a discrete integration by parts

$$\sum_{\omega\in(\lambda\Lambda_N)^d} \lambda^d \chi(\omega) \underline{\Delta}_\lambda e_\omega(x) = \sum_{\omega\in(\lambda\Lambda_N)^d} \lambda^d \underline{\Delta}_\lambda \chi(\omega) e_\omega(x)\ .$$

Repeating the above procedure and using the fact that $\cos(\pi\lambda x_i) - 1 \leq -c_0\lambda^2|x_i|^2$ for every $|x| \leq \lambda^{-1}$

$$|\check{\chi}_\lambda(x)| \leq \left(\lambda^{-2}\sum_{i=1}^{d} 1 - \cos(\pi\lambda x_i)\right)^{-k} \left|\sum_{\omega\in(\lambda\Lambda_N)^d} \lambda^d \underline{\Delta}_\lambda^k \chi(\omega) e_\omega(x)\right|$$
$$\leq c_0^k|x|^{-2k} \sum_{\omega\in(\lambda\mathbb{Z})^d} \lambda^d |\underline{\Delta}_\lambda^k \chi(\omega)| \leq C(k)|x|^{-2k}$$

where in the last inequality we used the fact that the Riemann sum is bounded for small λ by $\left\|\Delta^k\chi\right\|_{L^1(\mathbb{R}^d)}$ which is finite since χ is smooth and compactly supported. □

We quote in the next proposition a useful embedding between Besov and L^p spaces proven, for instance, in [BCD11, Prop. 2.39].

Proposition A.0.3 *For any $\nu > 0$, and $p \geq \frac{d}{\nu}$ there exists $C > 0$*

$$\|f\|_{\mathcal{B}^{-\nu}_{\infty,\infty}(\mathbb{T}^d)} \leq C \|f\|_{L^p(\mathbb{T}^d)}$$

We now prove some estimates that are difficult to find in the context of discrete Besov spaces. Their proofs follows closely their continuous counterparts that can be found, for instance, in [BCD11, Chapter 2].

Proposition A.0.4 (Duality for discrete Besov spaces) *Let $\alpha \in \mathbb{R}$, $p, q, p', q' \geq 1$ with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. For $f, g : \Lambda_\varepsilon \to \mathbb{R}$, there exists $C = C(\alpha) > 0$*

$$\langle f, g\rangle_{\Lambda_\varepsilon} \leq C \|f\|_{\mathcal{B}^\alpha_{p,q}(\Lambda_\varepsilon)} \|g\|_{\mathcal{B}^{-\alpha}_{p',q'}(\Lambda_\varepsilon)}$$

Proof. Use the Plancherel theorem and the Paley-Littlewood decomposition to write

$$\langle f, g\rangle_{\Lambda_\varepsilon} = \sum_{\omega\in\Lambda_N} \hat{f}(\omega)\overline{\hat{g}(\omega)}$$
$$= \sum_{k,k'\geq -1}\sum_{\omega\in\Lambda_N} \widehat{\eta_k}(\omega)\hat{f}(\omega)\overline{\widehat{\eta_{k'}}(\omega)\hat{g}(\omega)} = \sum_{|k-k'|\leq 1} \langle \eta_k *_\epsilon f, \eta_{k'} *_\epsilon g\rangle_{\Lambda_\varepsilon}$$

where in the last equality we used the fact that $\chi_k(\omega)\chi_{k'}(\omega) = 0$ for $|k - k'| > 1$. The proposition is proven using

$$|\langle \eta_k *_\epsilon f, \eta_{k'} *_\epsilon g\rangle_{\Lambda_\varepsilon}| \leq 2^{|\alpha|}2^{-\alpha k} \|\eta_k *_\epsilon f\|_{L^p(\Lambda_\varepsilon)} 2^{\alpha k'} \|\eta_{k'} *_\epsilon g\|_{L^{p'}(\Lambda_\varepsilon)}$$

and the Hölder inequality. □

Proposition A.0.5 (Multiplicative inequality) *Let $a, b > 0$ with $a < b$. Assume f to be in $\mathcal{C}^{-a}(\mathbb{T}^d)$ and g to be in $\mathcal{C}^b(\mathbb{T}^d)$. Then the pointwise product fg (well defined on a dense subspace of $\mathcal{C}^{-a}(\mathbb{T}^d)$) can be extended to a bilinear continuous map $\mathcal{C}^{-a}(\mathbb{T}^d) \times \mathcal{C}^b(\mathbb{T}^d) \to \mathcal{C}^{-a}(\mathbb{T}^d)$ and*

$$\|fg\|_{\mathcal{C}^{-a}(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{C}^{-a}(\mathbb{T}^d)} \|g\|_{\mathcal{C}^b(\mathbb{T}^d)} \ . \tag{A.7}$$

Proposition A.0.6 (Product estimates for discrete Besov spaces) *Let $\beta < 0 < \alpha$ and $p, q \in [1, \infty]$. For $f, g : \Lambda_\varepsilon \to \mathbb{R}$, there exists $C > 0$ such that*

$$\|fg\|_{\mathcal{C}^\beta(\Lambda_\varepsilon)} \leq C \|f\|_{\mathcal{C}^\beta(\Lambda_\varepsilon)} \|g\|_{\mathcal{C}^\alpha(\Lambda_\varepsilon)}$$

Proof. Recall the fact that on Λ_ε, we have $\mathsf{Ext}(fg)(x) = \mathsf{Ext}(f)(x)\mathsf{Ext}(g)(x)$ since the extension operator coincide with the identity on points of Λ_ε. The proof of the proposition relies on the paraproduct decomposition

$$uv = T_u v + T_v u + R(u, v)$$

where the operators are defined

$$\begin{aligned} T_u v &\stackrel{\text{def}}{=} \sum_{-1 \leq k < k'-1} \delta_k u \, \delta_{k'} v = \sum_{k' \geq -1} S_{k'-1} u \, \delta_{k'} v \\ R(u, v) &\stackrel{\text{def}}{=} \sum_{|k-k'| \leq 1} \delta_k u \, \delta_{k'} v \end{aligned}$$

In [BCD11, Sec. 2.6] it is proven that for $p_1, p_2, r_1, r_2 \in [1, \infty]$, let

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1 \ .$$

- If $s \in \mathbb{R}$, $t < 0$. Then the operator T can be extended to a continuous bilinear map $\mathcal{B}^t_{p_1,r_1}(\mathbb{T}^d) \times \mathcal{B}^s_{p_2,r_2}(\mathbb{T}^d)$ to $\mathcal{B}^t_{p,r}(\mathbb{T}^d)$
- If $s, t \in \mathbb{R}$ with $s + t > 0$. Then the operator R can be extended to a continuous bilinear map $\mathcal{B}^t_{p_1,r_1}(\mathbb{T}^d) \times \mathcal{B}^s_{p_2,r_2}(\mathbb{T}^d)$ to $\mathcal{B}^{t+s}_{p,r}(\mathbb{T}^d)$.

We are going to prove a similar result in the discrete setting, using the proof in [MW17b, Cor. 3.1]. We will show for $s, t \in \mathbb{R}$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$, for $p_1, p_2, r_1, r_2 \in [1, \infty]$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \qquad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$$

that there exists C, independent of ϵ, u, v such that

$$\|T_u v\|_{\mathcal{B}^{(t\wedge 0)+s}_{p,r}(\Lambda_\varepsilon)} \leq C \|u\|_{\mathcal{B}^{t}_{p_1,\infty}(\Lambda_\varepsilon)} \|v\|_{\mathcal{B}^{s}_{p_2,r}(\Lambda_\varepsilon)}$$
$$\|R(u,v)\|_{\mathcal{B}^{s_1+s_2}_{p,r}(\Lambda_\varepsilon)} \leq C \|u\|_{\mathcal{B}^{s_1}_{p_1,r_1}(\Lambda_\varepsilon)} \|v\|_{\mathcal{B}^{s_2}_{p_2,r_2}(\Lambda_\varepsilon)} \; .$$

Apply the inequality to $\mathsf{Ext}(f)$ and $\mathsf{Ext}(g)$ when $p = p_1 = p_2 = r = r_1 = r_2 = \infty$ shows the result.

The above inequalities follow essentially from the application of the discrete Hölder inequality instead of the continuous one. Recall the fact that the continuous Fourier transform of the product $\mathsf{Ext}(f)\mathsf{Ext}(g)$ has non zero frequencies for $\epsilon^{-1} < |\omega|_\infty \leq 2\epsilon^{-1}$. Those frequencies are contained only in the diagonal term $R(\mathsf{Ext}(f), \mathsf{Ext}(g))$.

We will first prove the bound for the operator T. For $\omega \sim 2^{-k}$ the frequencies $\widehat{T_u v}(\omega)$ are coming from the product $S_{k'-1}u\ \delta_{k'}v$ for $|k-k'| \leq 2$. Therefore the Besov norm $\|T_u v\|_{\mathcal{B}^{t}_{p,r}(\Lambda_\varepsilon)}$ is equivalent to

$$\|T_u v\|_{\mathcal{B}^{(0\wedge t)+s}_{p,r}(\Lambda_\varepsilon)} \lesssim \left\| \left(2^{k(0\wedge t)} 2^{ks} \|S_{k-1}u\ \delta_k v\|_{L^p(\Lambda_\varepsilon)} \right)_{k\geq 0} \right\|_{l^r}$$

and

$$\|S_{k-1}u\ \delta_k v\|_{L^p(\Lambda_\varepsilon)} \leq \|S_{k-1}u\|_{L^{p_1}(\Lambda_\varepsilon)} \|\delta_k v\|_{L^{p_2}(\Lambda_\varepsilon)}$$

therefore using the fact that $\sum_{k=1}^N a^k \lesssim a^N \vee 1$

$$\|S_{k-1}u\|_{L^{p_1}(\Lambda_\varepsilon)} \leq \sum_{-1\leq k'<k-1} \|\delta_{k'}u\|_{L^{p_1}(\Lambda_\varepsilon)} \lesssim 2^{-(0\wedge t)k} \sup_{-1\leq k'<k-1} 2^{tk'} \|\delta_{k'}u\|_{L^{p_1}(\Lambda_\varepsilon)}$$

and therefore $\|T_u v\|_{\mathcal{B}^{t}_{p,r}(\Lambda_\varepsilon)}$is bounded by

$$\sup_{-1\leq k'} 2^{tk'} \|\delta_{k'}u\|_{L^{p_1}(\Lambda_\varepsilon)} \left\| \left(2^{ks} \|\delta_k v\|_{L^{p_2}(\Lambda_\varepsilon)} \right)_{k\geq 0} \right\|_{l^r} \leq \|u\|_{\mathcal{B}^{t}_{p_1,\infty}(\Lambda_\varepsilon)} \|v\|_{\mathcal{B}^{s}_{p_2,r}(\Lambda_\varepsilon)}$$

In order to bound $\|R(u,v)\|_{\mathcal{B}^{s_1+s_2}_{p,r}(\Lambda_\varepsilon)}$ when $s_1 + s_2 > 0$, we will use the fact that if the function $F : \Lambda_\varepsilon \to \mathbb{R}$ can be written as the sum $F = \sum_{k\geq -1} F_k$, where for each $j \geq -1$, $F_j : \Lambda_\varepsilon \to \mathbb{R}$ has Fourier transform supported on a ball of radius 2^j, then

$$\|F\|_{\mathcal{B}^{s}_{p,r}(\Lambda_\varepsilon)} \lesssim \left\| \left(2^{sk} \|F_j\|_{L^p(\Lambda_\varepsilon)} \right)_{k\geq -1} \right\|_{l^r} \; .$$

This is proven in [BCD11, Lemma 2.49] for the usual Besov spaces, the extension to the discrete case being straightforward. In particular the Fourier transform of the product

$\delta_k u\, \delta_{k'} v$ is supported on a ball of radius proportional to 2^{2k} when $|k-k'| \le 1$. Then

$$\|F\|_{\mathcal{B}^{s_1+s_2}_{p,r}(\Lambda_\varepsilon)} \lesssim \left\| \left(2^{s_1 k} 2^{s_2 k} \sum_{k':|k-k'|\le 1} \|\delta_k u\, \delta_{k'} v\|_{L^p(\Lambda_\varepsilon)} \right)_{k\ge -1} \right\|_{l^r}$$

and two applications of the Hölder inequality are sufficient to prove the result. □

As the notation suggests, the continuous and discrete Besov norms are not so different. The next proposition shows a relationship between them.

Proposition A.0.7 *For $\beta \in \mathbb{R}$, and $p, q \in [1,\infty]$, there exist a constant $C > 0$ such that for any ϵ small enough and any function $f : \Lambda_\varepsilon^d \to \mathbb{R}$*

$$\|f\|_{\mathcal{B}^\beta_{p,q}(\Lambda_\varepsilon)} \le C \|f\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)}$$

Proof. We will first prove that, for all $k \ge -1$ we have

$$\Big| \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon)} - \|\delta_k \mathsf{Ext}(f)\|_{L^p(\mathbb{T}^d)} \Big| \le C\epsilon \|\nabla \delta_k \mathsf{Ext}(f)\|_{L^p(\mathbb{T})} \;.$$

This is similar to the calculation in Lemma A.0.1 (here the value of the constant might change from line to line)

$$\begin{aligned}
\Big| \|\delta_k \mathsf{Ext}(f)\|_{L^p(\Lambda_\varepsilon)} - \|\delta_k \mathsf{Ext}(f)\|_{L^p(\mathbb{T}^d)} \Big|^p &\le \Big| \|\delta_k \mathsf{Ext}(f)\|^p_{L^p(\mathbb{T}^d)} - \|\delta_k \mathsf{Ext}(f)\|^p_{L^p(\Lambda_\varepsilon)} \Big| \\
&\le C \sum_{x\in\Lambda_\varepsilon} \int_{|y|\le \epsilon/2} |\delta_k(\mathsf{Ext} f(x+y) - \mathsf{Ext} f(x))|^p dy \\
&\le C \sum_{x\in\Lambda_\varepsilon} \int_{|y|\le \epsilon/2} \left| \int_0^1 y \cdot \nabla \delta_k \mathsf{Ext} f(x+yt) dt \right|^p dy \\
&\le C\epsilon^p \int_{y\in\mathbb{T}^d} |\nabla \delta_k \mathsf{Ext} f(y)|^p \, dy \,.
\end{aligned}$$

If we sum up, for $k \ge -1$, we obtain

$$\Big| \|f\|_{\mathcal{B}^\beta_{p,q}(\Lambda_\varepsilon)} - \|f\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)} \Big| \le C\epsilon \|Df\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)}$$

Using the Bernstein's lemma and the fact that $2^k \lesssim \epsilon^{-1}$ concludes the proof

$$\epsilon \|Df\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)} \le C\epsilon \|f\|_{\mathcal{B}^{\beta+1}_{p,q}(\mathbb{T}^d)} \le C \|f\|_{\mathcal{B}^\beta_{p,q}(\mathbb{T}^d)} \;.$$

□

If the above proposition is stating that the continuous Besov norm controls the discrete one,

it seems unlikely for the reverse to hold as well.
However, we will show that the two norms are close if the function has a better regularity.

Corollary A.0.8 *For any $\lambda \in [0,1]$, there exists a $C = C(\beta, p, q, \lambda) > 0$ such that*

$$\left| \|f\|_{\mathcal{B}^{\beta}_{p,q}(\Lambda_\varepsilon)} - \|f\|_{\mathcal{B}^{\beta}_{p,q}(\mathbb{T}^d)} \right| \leq C\epsilon^{1-\lambda} \|f\|_{\mathcal{B}^{\beta+1-\lambda}_{p,q}(\mathbb{T}^d)} \tag{A.8}$$

Despite the fact that it is not needed in the present work, we will state a proposition that correspond to the discrete version of [MW17a, Prop. 3.25].

Proposition A.0.9 *For $f : \Lambda_\varepsilon \to \mathbb{R}$ and $\nu \in (0,1)$ and recall the definition of the discrete gradient $\nabla_\epsilon f$, one has*

$$\|f\|_{\mathcal{B}^{\nu}_{1,1}(\Lambda_\varepsilon)} \lesssim \|f\|_{L^1(\Lambda_\varepsilon)} + \|f\|^{1-\nu}_{L^1(\Lambda_\varepsilon)} \|\nabla_\epsilon f\|^{\nu}_{L^1(\Lambda_\varepsilon;\mathbb{R}^d)} \tag{A.9}$$

We will not prove the proposition, because the proof is identical to the one of Lemma A.0.11.

Remark A.0.10 We could have easily avoided the definition of a discrete version of the Besov norm. Indeed the Paley-Littlewood projections δ_k are well defined also for functions supported on Λ^d_ε, and also in this case $\delta_k f : \mathbb{T}^d \to \mathbb{R}$ is defined on $\mathbb{T}^d$. There is only one point where such definition is really needed, and this is in Proposition 3.3.3 and Lemma A.0.11, where in a technical point we need to control the Besov norm $\|f\|_{\mathcal{B}^{\nu}_{1,1}(\Lambda_\varepsilon)}$ with a particular discrete, long-range derivative of f. In that case, the use of Lemma A.0.1, Proposition A.0.7 or Proposition A.0.9 doesn't seem to be sufficient. Because of the discrete nature of the inequality (A.10), we were only able to show it for the Discrete Besov norm.

The next lemma is the main innovative tool of Chapter 2, and it allows to control the discrete Besov norm with the same discrete Laplacian arising from the Glauber dynamic. I am thankful to Martin Hairer for having simplified the argument of the proof.

Lemma A.0.11 *For $d = 2$ and $f : \Lambda_\varepsilon \to \mathbb{R}$ and $\nu \in (0, 1/2)$*

$$\|f\|_{\mathcal{B}^{\nu}_{1,1}(\Lambda_\varepsilon)} \lesssim \|f\|^{1-2\nu}_{L^1(\Lambda_\varepsilon)} \left(\sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y)\epsilon^{-1}\gamma|f(x) - f(y)| \right)^{2\nu} + \|f\|_{L^1(\Lambda_\varepsilon)} \tag{A.10}$$

where the constant is independent of ϵ or f.

The reader is encouraged to compare (A.10) with the same inequality in case of a continuous Laplacian (A.9). The factor 2 in front of ν depends on the scale at which Δ_γ

changes its behaviour, and this is not the best result that is possible to obtain. It is easy to extend the proof in any dimension: in this case one has to replace the exponent 2ν with $\frac{\log(\epsilon)}{\log(\gamma)}\nu$.

Proof. Rewrite the definition of $\|f\|_{\mathcal{B}^{\nu}_{1,1}(\Lambda_\varepsilon)}$ in (A.4) as

$$\|f\|_{\mathcal{B}^{\nu}_{1,1}(\Lambda_\varepsilon)} = \sum_{k\geq -1} 2^{\nu k} \|\eta_k * f\|_{L^1(\Lambda_\varepsilon)} \tag{A.11}$$

where η_k are the projections on the Paley-Littlewood blocks defined in (A.2). In the discrete case the summation over k extends up to a multiple of $\log(\epsilon^{-1})$. In the proof, since there is no possibility of confusion, we will use L^p instead of $L^p(\Lambda_\varepsilon)$. We will divide the sum into

$$\sum_{-1\leq k\leq L} 2^{\nu k} \|\eta_k * f\|_{L^1} + \sum_{L<k\leq -\log_2(\epsilon)} 2^{\nu k} \|\eta_k * f\|_{L^1}$$

where L will be chosen later. We bound the first part with

$$\sum_{-1\leq k\leq L} 2^{\nu k} \|\eta_k * f\|_{L^1} \leq \sum_{-1\leq k\leq L} 2^{\nu k} \sup_{k'\leq L} \|\eta_{k'}\|_{L^1} \|f\|_{L^1} \lesssim 2^{\nu L} \|f\|_{L^1} \ . \tag{A.12}$$

In order to control the second summation we will now prove, for $k \geq 0$, the inequality

$$\|\eta_k * f\|_{L^1} \lesssim \left(2^{-k} \vee \epsilon\gamma^{-1}\right) \sum_{x,y\in\Lambda_\varepsilon} \epsilon^4 K_\gamma(x-y) \frac{|f(y)-f(x)|}{\epsilon\gamma^{-1}} \ . \tag{A.13}$$

If $k \geq 0$ the projection kernel η_k has mean zero and therefore

$$\|\eta_k * f\|_{L^1} = \sum_{x\in\Lambda_\varepsilon} \epsilon^2 \left| \sum_{y\in\Lambda_\varepsilon} \epsilon^2 \eta_k(-y) \Big(f(x+y) - f(x)\Big) \right| \ .$$

At this point the treatment differs from the proof of [MW17b, Prop. 3.25], because of the particular form of the Laplacian. The definition of K_γ (in particular the continuity of $\mathfrak{K}$) implies that there exists $b_0 > 0$ such that

$$\inf_{|w|\leq b_0\epsilon\gamma^{-1}} \sum_{z\in\Lambda_\varepsilon} \epsilon^2 \big(K_\gamma(z) \wedge K_\gamma(w-z)\big) \geq 1/2 \, .$$

If $|y| \leq b_0\epsilon\gamma^{-1}$, then

$$\Big|f(x+y)-f(x)\Big| \leq 2\Big(\sum_{z\in\Lambda_\varepsilon}\epsilon^2\big(K_\gamma(z)\wedge K_\gamma(y-z)\big)\Big)\Big|f(x+y)-f(x)\Big|$$
$$\leq 2\epsilon^2\sum_{z\in\Lambda_\varepsilon}K_\gamma(y-z)\Big|f(x+y)-f(x+z)\Big|+K_\gamma(z)\Big|f(x+z)-f(x)\Big|\,.$$

If $|y| \geq b_0\epsilon\gamma^{-1}$ on the other hand, then there exists a path $\{y_0, y_1, \ldots, y_n\}$ in Λ_ε of length n proportional to $|y|\gamma\epsilon^{-1}$ connecting $y_0 = 0$ with $y_n = y$ and such that $|y_{j+1} - y_j| \leq b_0\epsilon\gamma^{-1}$ for $j = 0, \ldots, n-1$. We can then apply the above inequality to every step of the path. Combining these bounds, we obtain

$$\|\eta_k * f\|_{L^1} \lesssim \sum_{y\in\Lambda_\varepsilon}\epsilon^2|\eta_k(-y)|\{|y|\gamma\epsilon^{-1}\vee 1\}\sum_{x\in\Lambda_\varepsilon,z\in\Lambda_\varepsilon}\epsilon^4K_\gamma(z)|f(x+z)-f(x)|\,, \tag{A.14}$$

and (A.13) follows from the fact that

$$\sum_{y\in\Lambda_\varepsilon}\epsilon^2|\eta_k(y)|\{|y|\epsilon^{-1}\gamma\vee 1\} \lesssim \|\eta_k\|_{L^1}+\epsilon^{-1}\gamma 2^{-k}\sum_{y\in\Lambda_\varepsilon}2^k|y||\eta_k(y)| \lesssim 1\vee\epsilon^{-1}\gamma 2^{-k}\,.$$

Summing over k yields

$$\begin{aligned}&\sum_{L<k\leq\log_2(\epsilon^{-1})}2^{\nu k}\,\|\eta_k * f\|_{L^1}\\ &\lesssim \sum_{L<k\leq\log_2(\epsilon^{-1})}2^{\nu k}\{\epsilon\gamma^{-1}\vee 2^{-k}\}\sum_{x\in\Lambda_\varepsilon,z\in\Lambda_\varepsilon}\frac{\epsilon^4K_\gamma(z)}{\epsilon\gamma^{-1}}|f(x+z)-f(x)|\,.\end{aligned} \tag{A.15}$$

At this point we use the fact that $\epsilon\gamma^{-1}\vee 2^{-k} \leq 2^{-\frac{k}{2}}$ for $k \leq -\log_2(\epsilon) = -2\log_2(\epsilon\gamma^{-1})$ and, recalling (A.11) and (A.12), we obtain

$$\|f\|_{\mathcal{B}^\nu_{1,1}(\Lambda_\varepsilon)} \lesssim 2^{\nu L}\,\|f\|_{L^1}+2^{(\nu-\frac{1}{2})L}\sum_{x\in\Lambda_\varepsilon,z\in\Lambda_\varepsilon}\epsilon^4K_\gamma(z)\frac{|f(x+z)-f(x)|}{\epsilon\gamma^{-1}}\,.$$

The claim now follows by optimising this expression over L. (The second term in (A.10) comes from the fact that we had to impose $L > 1$.) □

The next lemma is a simple but crucial result and is proven in [TW16].

Lemma A.0.12 (Comparison test) *Let $\lambda > 1$ and $f : [0,T] \to \mathbb{R}^+$ a differentiable function satisfying for $t \in [0,T]$*

$$f'(t)+2c_1\,(f(t))^\lambda \leq c_2\,.$$

Then for $t \in [0, T]$

$$f(t) \leq \frac{f(0)}{(1 + c_1(\lambda - 1)tf(0)^{\lambda-1})^{\frac{1}{\lambda-1}}} \vee \left(\frac{c_2}{c_1}\right)^{\frac{1}{\lambda}} \leq (c_1(\lambda - 1)t)^{-\frac{1}{\lambda-1}} \vee \left(\frac{c_2}{c_1}\right)^{\frac{1}{\lambda}} .$$

Appendix B

Estimation on kernels and discrete semigroups

We collect, for reference, some of the estimates in [MW17a], some of which have been generalized to dimensions $d = 1, 2$ or adapted to our context. In this case we are providing a description of how the proof in [MW17a] should be modified to accommodate our situation. We will consider $d = 1, 2$ and recall the definitions introduced in Chapter 1 of $\Lambda_N = \{1 - N, N\}$. Define $\epsilon = N^{-1}$ and let $\Lambda_\varepsilon = \epsilon\Lambda_N$ be a discrete approximation of the torus $\mathbb{T}$. In this appendix we will collect bounds used in the previous chapters and therefore we will not assume any particular relation between γ and ϵ to hold, unless otherwise stated.

Consider the Fourier transform of the kernel K_γ, which is given, for $\omega \in \Lambda_N^d$ by

$$\hat{K}_\gamma(\omega) = \frac{1}{2^d} \sum_{x \in \Lambda_\varepsilon^d} \epsilon^d K_\gamma(x) e^{-\pi\omega \cdot x} = \frac{1}{2} \sum_{z \in \{1-N, N\}^d} \epsilon^d \gamma^{-2} \kappa_\gamma(\epsilon z) e^{-\pi\epsilon\omega \cdot z} \, .$$

For some of the estimates it will be useful to work with the partial derivative of $\hat{K}_\gamma(\omega)$ with respect to the j-th component of ω. In this case the derivative $\partial_j \hat{K}_\gamma(\omega)$ should be interpreted using the fact that the expression $e^{-\pi\epsilon\omega \cdot z}$ is well defined also for $\omega \in \mathbb{R}^d$.

Proposition B.0.1 *We have that, for $|\omega| \leq \gamma\epsilon^{-1}$ there exists a constant C*

$$|\hat{K}_\gamma(\omega)| \leq 1$$
$$|\partial_j \hat{K}_\gamma(\omega)| \leq C\epsilon^2\gamma^{-2}|\omega|$$
$$|\partial_j^2 \hat{K}_\gamma(\omega)| \leq C|\epsilon\gamma^{-1}|^2$$

for $|\omega| \geq \gamma\epsilon^{-1}$

$$\begin{aligned}|\hat{K}_\gamma(\omega)| &\leq C|\epsilon\gamma^{-1}\omega|^{-2}\\ |\partial_j\hat{K}_\gamma(\omega)| &\leq C\epsilon\gamma^{-1}|\epsilon\gamma^{-1}\omega|^{-2}\\ |\partial_j^2\hat{K}_\gamma(\omega)| &\leq C\epsilon^2\gamma^{-2}|\epsilon\gamma^{-1}\omega|^{-2}\ .\end{aligned}$$

Moreover for any $|\omega| \leq \epsilon^{-1}$ *there exists a constant* $c > 0$

$$1 - \hat{K}_\gamma(\omega) \geq c|\epsilon\gamma^{-1}\omega|^2\ .$$

A proof of this proposition is given in [MW17a, Lemma 8.2]. The next proposition correspond to [MW17a, Lemma 8.1] and states that Δ_γ is a good approximation of the Laplacian for small Fourier modes.

Proposition B.0.2 *There exists a* $C > 0$ *such that for* γ *small enough,* $|\omega| \leq \epsilon^{-1}\gamma$ *and* $j \in \{1, \ldots, d\}$

$$\begin{aligned}\left|\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega)) - \pi^2|\omega|^2\right| &\leq C\epsilon\gamma^{-1}|\omega|^3\ ,\\ \left|-\epsilon^{-2}\gamma^2\partial_j\hat{K}_\gamma(\omega) - 2\pi^2\omega_j\right| &\leq C\epsilon\gamma^{-1}|\omega|^2\ ,\\ \left|-\epsilon^{-2}\gamma^2\partial_j^2\hat{K}_\gamma(\omega) - 2\pi^2\right| &\leq C\epsilon\gamma^{-1}|\omega|\ .\end{aligned}$$

The next lemma provides an estimate of the $L^\infty(\mathbb{T}^2)$-norm for $P_t^\gamma K_\gamma$. The proof is given in [MW17a, Lemma 8.3]

Lemma B.0.3 *For* $T > 0$*,* $x \in \mathbb{T}^2$ *there exists a* $C = C(T)$ *we have*

$$|P_t^\gamma K_\gamma(x)| \leq C(t^{-1} \wedge \epsilon^{-2}\gamma^2)\log\gamma^{-1}\ . \tag{B.1}$$

Lemma B.0.4 *For* γ *small enough*

$$\sup_{t\geq 0}\int_0^t \sum_{\omega\in\Lambda_\varepsilon^d} |\widehat{P_{t-s}^\gamma K}_\gamma(\omega)|^2 ds \leq \frac{1}{2}\sum_{0<|\omega|\leq\epsilon^{-1}} \frac{|\hat{K}_\gamma(\omega)|^2}{\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))} \lesssim \log(\gamma^{-1})\ . \tag{B.2}$$

The lemma follows immediately from the estimations in Proposition B.0.1.

Lemma B.0.5 *For any* $\nu > 0$ *and* $\kappa > 0$ *there exists constants* $c, C(\nu)$ *such that for all* $X : \mathbb{T}^2 \to \mathbb{R}$ *for which* $\hat{X}(\omega) = 0$ *for all* $|\omega| > \epsilon^{-2}$ *we have*

$$\|X\|_{L^\infty(\mathbb{T}^2)} \leq C(\nu)\epsilon^{-\nu}\|X\|_{\mathcal{C}^{-\nu}}\ .$$

The proof of the above proposition is given in the Appendix of [MW17a].

We recall now some bounds from [MW17a, Sec 8] regarding the semigroup associated to the diffusion.
Recall that, from [BCD11, Lemma 2.4], for the heat semigroup P_t and an element X of $\mathcal{C}^\nu$ we have for $\beta > 0$

$$\|P_t X\|_{\mathcal{C}^{\nu+\beta}} \leq C(\nu,\beta) t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu} \ . \tag{B.3}$$

The next proposition will provide bounds for the approximate heat semigroup P_t^γ similar to (B.3).

Proposition B.0.6 *For γ sufficiently small, for $c_1, c_2 > 0$, $T > 0$, $\kappa > 0$.*
Then

- *For $\beta > 0$ and $0 \leq \lambda \leq 1$ there exists $C = C(c_1, T, \kappa, \beta, \lambda)$ such that for all functions $X : \mathbb{T}^2 \to \mathbb{R}$ with $\hat{X}(\omega) = 0$ for all $|\omega| \geq c_1 \epsilon^{-1}\gamma$ we have that for all $t \in [0,T]$ and $\nu \in \mathbb{R}$*

$$\begin{aligned}
\|P_t^\gamma X\|_{\mathcal{C}^{\nu+\beta-\kappa}} &\leq C t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu} \\
\|(P_t^\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} &\leq C\gamma^\lambda \left(t^{-\frac{\lambda}{2}} \|X\|_{\mathcal{C}^\nu} \wedge \|X\|_{\mathcal{C}^{\nu+\lambda}} \right) \\
\|K_\gamma \star X\|_{\mathcal{C}^{\nu-\kappa}} &\leq C \|X\|_{\mathcal{C}^\nu} \\
\|K_\gamma \star X - X\|_{\mathcal{C}^{\nu-\kappa}} &\leq C\gamma^{2\lambda} \|X\|_{\mathcal{C}^{\nu+2\lambda}}
\end{aligned}$$

- *Let n be such that $\epsilon = \gamma^n$. For $\beta > 0$ and $\lambda > 0$ there exists $C = C(c_2, T, \kappa, \beta, \lambda)$ such that for any distribution X with $\hat{X}(\omega) = 0$ for $|\omega| \leq c_2 \epsilon^{-1}\gamma$, for $t \in [0,T]$ and $\nu \in \mathbb{R}$*

$$\|P_t^\gamma X\|_{\mathcal{C}^{\nu+\beta-\kappa}} \leq C t^{-\beta\frac{n}{2(n-1)} - \lambda} (\epsilon\gamma^{-1})^\lambda \|X\|_{\mathcal{C}^\nu} \tag{B.4}$$

  and if $0 \leq \beta \leq 2$

$$\|P_t^\gamma K_\gamma \star X\|_{\mathcal{C}^{\nu+\beta-\kappa}} \leq C t^{-\frac{\beta}{2}} \|X\|_{\mathcal{C}^\nu} \ .$$

This corresponds to lemma 8.4 in [MW17a], we highlight a small difference in (B.4), due to the different scaling. The bound produced is actually better for higher values of n.
A proof of (B.4) is given using the inequality, valid for $\epsilon^{-1}\gamma \leq |\omega| \leq \epsilon^{-1}$

$$e^{-t\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))} \leq \exp\left(-\frac{t}{C_1}\gamma^{2-2n} \right) \lesssim t^{-\beta\frac{n}{2(n-1)}} \gamma^{2\beta n} \lesssim t^{-\beta\frac{n}{2(n-1)}} |\omega|^{-\beta} \ .$$

The proof of the next proposition is given in [MW17a, Cor. 8.7].

Proposition B.0.7 *Let n be such that $\epsilon = \gamma^n$. For γ small enough, for $T > 0$, $\kappa > 0$ and $0 \le \lambda \le 1$ there exists a constant $C = C(T, \kappa, \beta, \lambda)$ such that for $t \in [0, T]$, $\nu \in \mathbb{R}$ and any distribution X on $\mathbb{T}^2$.*

$$\|(P_t^\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} \le C(\epsilon\gamma^{-1})^\lambda \left(t^{-\frac{n}{2(n-1)}\lambda} \|X\|_{\mathcal{C}^\nu} \wedge \|X\|_{\mathcal{C}^{\nu+\lambda}}\right)$$
$$\|(P_t^\gamma K_\gamma - P_t)X\|_{\mathcal{C}^{\nu-\kappa}} \le C(\epsilon\gamma^{-1})^\lambda \left(t^{-\frac{\lambda}{2}} \|X\|_{\mathcal{C}^\nu} \wedge \|X\|_{\mathcal{C}^{\nu+\lambda}}\right) .$$

The next lemma will be used in Section 2.5. It is proven in the same way as the above propositions.

Lemma B.0.8 *For $0 < \lambda$ and any $\kappa > 0$ there exists a constant $C = C(T, \lambda, \kappa)$ such that for $0 \le t \le T$ such that*

$$\|P_t - P_t^\gamma K_\gamma\|_{\mathcal{C}^{-\nu} \to \mathcal{C}^\beta} \le C(\epsilon\gamma^{-1})^{2\lambda} t^{-\lambda - \frac{\nu+\beta}{2} - \kappa} . \tag{B.5}$$

B.1 Semigroup associated to the Cahn-Hilliard equation

Let P_t^K be the semigroup associated to the fourth-order linear equation

$$\partial_t z = -\Delta^2 z .$$

In particular P_t^K is more conveniently described as a pseudo-differential operator via its Fourier transform $\hat{P}_t^K(\omega) = e^{-\pi^4 t|\omega|^4}$. For the operator P_t^K a smoothing effect, similar to (B.3) holds.

Lemma B.1.1 *Consider an annulus $\mathcal{A} = \{r \in \mathbb{R}^d : c_0 \le |r| \le c_1\}$ for some constants $0 < c_0 < c_1$. Then, there exists positive constants C, c such that, for all $\lambda > 0$ and $f : \mathbb{T}^d \to \mathbb{R}$ smooth functions whose Fourier transform is supported on the annulus $\lambda\mathcal{A}$ we have*

$$\left\|P_t^K f\right\|_{L^p(\mathbb{T}^d)} \le C e^{-ct\lambda^4} \|f\|_{L^p(\mathbb{T}^d)}$$
$$\left\|P_t^K \Delta f\right\|_{L^p(\mathbb{T}^d)} \le C \lambda^2 e^{-ct\lambda^4} \|f\|_{L^p(\mathbb{T}^d)} .$$

As a corollary of the above inequality we have

Proposition B.1.2 (Smoothing effect of P_t^K) *For $\beta \ge 0$, $\nu \in \mathbb{R}$, and $p, q \in [0, \infty]$, there* *exists a constant $C = C(\beta, \nu, p, q) > 0$ such that*

$$\left\|P_t^K f\right\|_{\mathcal{B}_{p,q}^{\beta+\nu}} \leq C t^{-\frac{\beta}{4}} \|f\|_{\mathcal{B}_{p,q}^{\nu}} ,$$
$$\left\|P_t^K \Delta f\right\|_{\mathcal{B}_{p,q}^{\beta+\nu}} \leq C t^{-\frac{\beta+2}{4}} \|f\|_{\mathcal{B}_{p,q}^{\nu}} .$$

We omit the proofs of the above propositions, since they follows closely the arguments in [BCD11, Lemma 2.4] and [MW17b, Prop. 3.11] for the heat semigroup, and because we are going to prove in detail similar inequalities for their discrete counterparts.

We recall the definition of the kernel J_γ associated to the discrete nearest neighborhood Laplacian $\Delta_\epsilon X = \epsilon^{-2}\left(J_\gamma X - X\right)$ on Λ_ε^d and and the discrete semigroup $P_t^{K,\gamma}$

$$\hat{P}_t^{K,\gamma}(\omega) = \exp\left\{-t\epsilon^{-2}(1-\hat{J}_\gamma(\omega))\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))\right\} . \tag{B.6}$$

We are now going to present some useful proprieties of $P_t^{K,\gamma}$ used in Chapter 4. For the following propositions we will make use of the representation

$$1-\hat{J}_\gamma(\omega) = \frac{1}{2^d}\sum_{e\in\mathbb{Z}^d:|e|_\infty=1}(1-\cos(\pi\epsilon e\cdot\omega)) = \frac{1}{2}\pi^2\epsilon^2|\omega|^2 + \mathcal{O}(\epsilon^4|\omega|^4)$$

and the fact that for some constants $0 < c < C$ we have that $c|\omega|^2 \leq 1-\hat{J}_\gamma(\omega) \leq C|\omega|^2$ for all $\omega \in \Lambda_N^d$.

Proposition B.1.3 *From the expression* (B.6)*, we see that the $\hat{P}_t^{K,\gamma}(\omega)$ has a meaningful extension for $\omega \in \mathbb{R}^d$. For such extension, denoted with $\hat{P}_t^{K,\gamma}$ as well, let $\partial_j\hat{P}_t^{K,\gamma}$ the continuous derivative the with respect to the j-th component of $\omega \in \mathbb{R}^d$. Then,*

- *For $|\omega| \leq \epsilon^{-1}\gamma$*

$$\hat{P}_t^{K,\gamma}(\omega) = \exp\left\{-t\left(\frac{\hat{K}_\gamma(\omega)-1}{\gamma^{-2}\epsilon^2}\right)\left(\frac{\hat{J}_\gamma(\omega)-1}{\epsilon^2}\right)\right\} \leq \exp(-ct|\omega|^4)$$
$$\begin{aligned}\partial_j\hat{P}_t^{K,\gamma}(\omega) &= \frac{\gamma^2}{\epsilon^4}\hat{P}_t^{K,\gamma}(\omega)\left(\partial_j\hat{K}_\gamma(\omega)(\hat{J}_\gamma(\omega)-1)+\partial_j\hat{J}_\gamma(\omega)(\hat{K}_\gamma(\omega)-1)\right)\\ &\lesssim \sqrt{t}\hat{P}_t^{K,\gamma}(\omega)\sqrt{t}\left(|\omega|^3+O(\epsilon^2)\right)\\ \partial_j^2\hat{P}_t^{K,\gamma}(\omega) &\lesssim t^2\hat{P}_t^{K,\gamma}(\omega)|\omega|^6\end{aligned}$$

- *For* $|\omega| \geq \epsilon^{-1}\gamma$

$$
\begin{aligned}
\hat{P}_t^{K,\gamma}(\omega) &= \exp\Big(-t\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))\epsilon^{-2}(1-\hat{J}_\gamma(\omega))\Big) \\
&\leq \exp\left(-ct\epsilon^{-2}\gamma^2|\omega|^2\right) \\
\partial_j\hat{P}_t^{K,\gamma}(\omega) &= \frac{\gamma^2}{\epsilon^4}t\hat{P}_t^{K,\gamma}(\omega)\Big(\partial_j\hat{K}_\gamma(\omega)(\hat{J}_\gamma(\omega)-1)+\partial_j\hat{J}_\gamma(\omega)(\hat{K}_\gamma(\omega)-1)\Big) \\
&\lesssim t\hat{P}_t^{K,\gamma}(\omega)(|\omega|+\epsilon^{-1}\gamma)\epsilon^{-2}\gamma^2 \lesssim t\hat{P}_t^{K,\gamma}(\omega)|\omega|\epsilon^{-2}\gamma^2 \\
\partial_j^2\hat{P}_t^{K,\gamma}(\omega) &= \frac{\gamma^2}{\epsilon^4}t^2\hat{P}_t^{K,\gamma}(\omega)\Big(\partial_j^2\hat{K}_\gamma(1-\hat{J}_\gamma)+2\partial_j\hat{K}_\gamma\partial_j\hat{J}_\gamma+(1-\hat{K}_\gamma)\partial_j^2\hat{J}_\gamma\Big) \\
&\lesssim t^2\hat{P}_t^{K,\gamma}(\omega)(\epsilon^{-2}\gamma^2+\epsilon^{-3}\gamma^3|\omega|^{-1})
\end{aligned}
$$

  where in the last equation we used $\hat{K}_\gamma$ *and* $\hat{J}_\gamma$ *instead of* $\hat{K}_\gamma(\omega)$ *and* $\hat{J}_\gamma(\omega)$. *Moreover*

$$
\begin{aligned}
&|\hat{P}_t^{K,\gamma}(\omega)\hat{\Delta}_\epsilon(\omega)\hat{K}_\gamma(\omega)| \leq Ce^{-ct\epsilon^{-2}\gamma|\omega|^2}\epsilon^{-2}\gamma^2 \\
&\leq \begin{cases} C_\beta t^{-\beta/4}(\epsilon^{-2}\gamma^2)^{1-\beta/4}|\omega|^{-\beta/2} \leq C_\beta t^{-\beta/4}|\omega|^{-(\beta-2)} & \textit{for } \beta\in[0,4] \\ C_\beta t^{-\beta/4-\frac{1}{2}}(\epsilon^{-2}\gamma^2)^{1-\beta/4-\frac{1}{2}}|\omega|^{-\beta/2-1} \leq C_\beta t^{-\beta/4-\frac{1}{2}}|\omega|^{-\beta} & \textit{for } \beta\in[0,2] \end{cases}
\end{aligned}
$$

  for some constant C_β.

- *For the scaling* (4.21) *in Chapter 4 and* $\epsilon^{-2}\gamma^2 = \epsilon^{-d/2} \geq |\omega|^{d/2}$ *and therefore there exists* $c > 0$ *such that*

$$
\left|\hat{P}_t^{K,\gamma}(\omega)\right| \leq \exp\Big(-ct|\omega|^{2+d/2}\Big)\ .
$$

The estimates from Proposition B.1.3 are the main ingredient of the following proposition, which contains the regularity improvement of the discrete semigroup, in terms of operator norms. This is in part the generalization of Lemma 4.3.5 and shows that the smoothing effect of the discrete semigroup $P_t^{K,\gamma}$ is essentially the same of its continuous counterpart.

Lemma B.1.4 *For all* $\kappa > 0$ *for* $p \in [1,2]$, *there exists* $C(\kappa,p) > 0$ *such that*

$$
\left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^p(\mathbb{T})\to L^\infty(\mathbb{T})} \leq C(\kappa)t^{-\frac{1}{4}\left(2+\frac{1}{p}+\frac{\kappa}{p}\right)}\ .
$$

Proof. Let q be such that $1 = \frac{1}{p}+\frac{1}{q}$. By the fact that $p\in[1,2]$ we have that $q\in[2,\infty]$ and by the convexity of L^p norms we have

$$\begin{aligned}\left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^p(\mathbb{T})\to L^\infty(\mathbb{T})} &= \left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^q(\mathbb{T})} \\ &\le \left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^2(\mathbb{T})}^{\frac{2}{q}} \left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^\infty(\mathbb{T})}^{1-\frac{2}{q}} \\ &= \left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^2(\mathbb{T})}^{2\left(1-\frac{1}{p}\right)} \left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^\infty(\mathbb{T})}^{-1+\frac{2}{p}}\end{aligned}$$

and therefore it is sufficient to prove the result for $q = 2$ and $q = \infty$. A quick calculation shows that

$$\begin{aligned}&\left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^2(\mathbb{T})}^2 \\ &\quad\le \sum_{|\omega|\le\epsilon^{-1}\gamma} |\omega|^4 e^{-ct|\omega|^4} + \sum_{\epsilon^{-1}\gamma\le|\omega|\le\epsilon^{-1}} (\epsilon^{-1}\gamma)^4 e^{-ct|\omega|^2\epsilon^{-2}\gamma^2} \\ &\quad\le C(\kappa)t^{-\frac{5+\kappa}{4}}\Big(\sum_{|\omega|\le\epsilon^{-1}\gamma}|\omega|^{-1-\kappa} + \sum_{|\omega|>\epsilon^{-1}\gamma}(\epsilon^{-1}\gamma)^{\frac{3-\kappa}{2}}|\omega|^{-\frac{5+\kappa}{2}}\Big) \le C(\kappa)t^{-\frac{5+\kappa}{4}}\end{aligned}$$

and in a similar way

$$\begin{aligned}\left\|(-\Delta_\epsilon)P_t^{K,\gamma}K_\gamma\right\|_{L^\infty(\mathbb{T})} &\lesssim \sum_{|\omega|\le\epsilon^{-1}} |\omega|^2|\hat{K}_\gamma(\omega)|\hat{P}_t^{K,\gamma}(\omega) \\ &\lesssim \sum_{|\omega|\le\epsilon^{-1}\gamma}|\omega|^2 e^{ct|\omega|^4} + \epsilon^{-2}\gamma^2 \sum_{\epsilon^{-1}\gamma\le|\omega|\le\epsilon^{-1}} e^{ct|\omega|^4} \le C(\kappa)t^{-\frac{3+\kappa}{4}}\ .\end{aligned}$$

This completes the proof of the lemma. □

Lemma B.1.5 *For any $\kappa > 0$ there exists $C(\kappa)$ such that for γ small enough*

$$\left\|\Delta P_{t-s}^K - \Delta_\epsilon P_{t-s}^{K,\gamma}K_\gamma\right\|_{L^2(\mathbb{T})\to L^\infty(\mathbb{T})} \le C(\kappa)(\epsilon\gamma^{-1})^{\frac{\kappa}{2}}t^{-\frac{5+\kappa}{4}}\ . \tag{B.7}$$

Proof. We will first show (B.7). We have that

$$\begin{aligned}&\left\|\Delta P_{t-s}^K - \Delta_\epsilon P_{t-s}^{K,\gamma}K_\gamma\right\|_{L^2(\mathbb{T})\to L^\infty(\mathbb{T})}^2 \\ &\lesssim \sum_{|\omega|\le\epsilon^{-1}}\Big(\pi^2|\omega|^2\hat{P}_{t-s}^K(\omega) - \hat{\Delta}_\epsilon(\omega)\hat{P}_{t-s}^{K,\gamma}(\omega)\hat{K}_\gamma(\omega)\Big)^2 + \sum_{|\omega|>\epsilon^{-1}}\Big(\pi^2|\omega|^2\hat{P}_{t-s}^K(\omega)\Big)^2\end{aligned}$$

for $|\omega| \le \epsilon^{-1}\gamma$ we have

$$\begin{aligned}\Big(\pi^2|\omega|^2\hat{P}^K_{t-s}(\omega) - &\hat{\Delta}_\epsilon(\omega)\hat{P}^{K,\gamma}_{t-s}(\omega)\hat{K}_\gamma(\omega)\Big)^2 \\ &\lesssim \left(\hat{P}^{K,\gamma}_{t-s}(\omega) - \hat{P}^K_{t-s}(\omega)\right)^2 |\omega|^4 \\ &+ \left(\pi^2|\omega|^2 - \epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))\right)^2 e^{-ct|\omega|^4} \\ &+ |\omega|^4 e^{-ct|\omega|^4}\left(1-\hat{K}_\gamma(\omega)\right)^2 .\end{aligned}$$

Using the definition of $\hat{P}^{K,\gamma}_t$ and Proposition B.0.2 we have

$$\begin{aligned}|\hat{P}^{K,\gamma}_t&(\omega) - \hat{P}^K_t(\omega)| \\ &\le e^{-tc|\omega|^4} t \left|\epsilon^{-2}\gamma^2(1-\hat{K}_\gamma(\omega))\epsilon^{-2}(1-\hat{J}_\gamma(\omega)) - \pi^4|\omega|^4\right| \\ &\le e^{-ct|\omega|^4} t|\omega|^4(\epsilon\gamma^{-1}|\omega| + \epsilon|\omega|)\end{aligned} \tag{B.8}$$

and using Proposition B.0.2 we have that

$$\begin{aligned}\sum_{|\omega|\le\epsilon^{-1}\gamma} &e^{-ct|\omega|^4}\left\{\epsilon^2\gamma^{-2}|\omega|^6 + \epsilon^2\gamma^{-2}t^2|\omega|^{14} + \epsilon^4\gamma^{-4}|\omega|^8\right\} \\ &\le C(\kappa)t^{-\frac{5+\kappa}{4}} \sum_{|\omega|\le\epsilon^{-1}\gamma} (\epsilon\gamma^{-1})^2|\omega|^{1-\kappa} + (\epsilon\gamma^{-1})^4|\omega|^{3-\kappa} \le C(\kappa)(\epsilon\gamma^{-1})^\kappa t^{-\frac{5+\kappa}{4}}\end{aligned}$$

for $\epsilon^{-1}\gamma \le |\omega| \le \epsilon^{-1}$ we can estimate separately the quantities

$$\begin{aligned}\sum_{\epsilon^{-1}\gamma\le|\omega|} &\left(\pi^2|\omega|^2\hat{P}^K_{t-s}(\omega)\right)^2 \le C \sum_{\epsilon^{-1}\gamma\le|\omega|} |\omega|^4 e^{-ct|\omega|^4} \\ &\le C \sum_{\epsilon^{-1}\gamma\le|\omega|} |\omega|^4 e^{-ct|\omega|^4} \le C(\kappa) \sum_{\epsilon^{-1}\gamma\le|\omega|} t^{-\frac{5+\kappa}{4}}|\omega|^{-1-\kappa} \le C(\kappa)t^{-\frac{5+\kappa}{4}}(\epsilon\gamma^{-1})^\kappa\end{aligned}$$

and

$$\begin{aligned}\sum_{\epsilon^{-1}\gamma\le|\omega|\le\epsilon^{-1}} &\left(\hat{\Delta}_\epsilon(\omega)\hat{P}^{K,\gamma}_{t-s}(\omega)\right)^2 \le C \sum_{\epsilon^{-1}\gamma\le|\omega|\le\epsilon^{-1}} \epsilon^{-4}\gamma^4|\hat{K}_\gamma(\omega)|^2 e^{-ct\epsilon^{-2}\gamma^2|\omega|^2} \\ \le C &\sum_{\epsilon^{-1}\gamma\le|\omega|\le\epsilon^{-1}} \frac{(\epsilon^{-1}\gamma)^8}{|\omega|^4} e^{-ct\epsilon^{-2}\gamma^2|\omega|^2} \le C(\kappa) \sum_{\epsilon^{-1}\gamma\le|\omega|} t^{-\frac{5+\kappa}{4}}(\epsilon^{-1}\gamma)^{\frac{11-\kappa}{2}}|\omega|^{-\frac{13+\kappa}{2}} \\ &\le C(\kappa)t^{-\frac{5+\kappa}{4}}(\epsilon\gamma^{-1})^\kappa .\end{aligned}$$

□

We are now going to prove some estimates used in Proposition 4.4.2. In the next proposition we assume $d = 1$, the generalization to more dimension being immediate.

Proposition B.1.6 *Let $d = 1$. For every $0 < \kappa < \frac{1}{2}$ there exists $C = C(\kappa)$ such that for all $0 \le s' \le s \le t$ and every $x, x' \in \Lambda_\varepsilon$ we have*

$$\int_{s'}^{s} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(z) \right)^2 dr \le C|s - s'|^{\frac{\kappa}{2}} , \tag{B.9}$$

$$\int_{0}^{s} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(z) - P^{K,\gamma}_{s-r} \nabla_\epsilon K_\gamma(z) \right)^2 dr \le C|t - s|^{\frac{\kappa}{2}} , \tag{B.10}$$

$$\int_{0}^{t} \sum_{z \in \Lambda_\varepsilon} \epsilon \left(P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x - z) - P^{K,\gamma}_{t-r} \nabla_\epsilon K_\gamma(x' - z) \right)^2 dr \le C|x - x'|^{2\kappa} . \tag{B.11}$$

Proof. The proof is a consequence of the definition (B.6) and the bounds of Proposition B.1.3. We are now going to show (B.9), the proof for (B.10), (B.11) being similar. By Plancherel theorem and Proposition B.1.3, for $\kappa < \frac{1}{2}$

$$\int_{s'}^{s} \sum_{\substack{\omega \in \mathbb{Z} \\ 0 < |\omega|_\infty \le \epsilon^{-1}\gamma}} |\omega|^2 e^{-c|\omega|^4 (t-r)} dr + \int_{s'}^{s} \sum_{0 < |\omega|_\infty \le \epsilon^{-1}\gamma} |\omega|^2 |\hat{K}_\gamma(\omega)|^2 e^{-c\epsilon^{-2}\gamma^2 |\omega|^2 (t-r)} dr$$

$$\lesssim \sum_{\substack{\omega \in \mathbb{Z} \\ 0 < |\omega|_\infty \le \epsilon^{-1}\gamma}} |\omega|^2 \frac{|\omega|^{4\kappa} |s - s'|^\kappa}{|\omega|^4} + \sum_{\substack{\omega \in \mathbb{Z} \\ \epsilon^{-1}\gamma < |\omega|_\infty \le \epsilon^{-1}}} |\omega|^2 \frac{(\epsilon^{-1}\gamma)^4}{|\omega|^4} \frac{(\epsilon^{-1}\gamma)^{2\kappa} |\omega|^{2\kappa} |s - s'|^\kappa}{\epsilon^{-2}\gamma^2 |\omega|^2}$$

$$\lesssim \left(1 + \epsilon\gamma^{-1}\right) |s - s'|^\kappa .$$

□

Bibliography

[Akh65] N. I. AKHIEZER. *The classical moment problem and some related questions in analysis*. Translated by N. Kemmer. Hafner Publishing Co., New York, 1965.

[Bax89] R. J. BAXTER. *Exactly solved models in statistical mechanics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1989. Reprint of the 1982 original.

[BCD11] H. BAHOURI, J.-Y. CHEMIN, and R. DANCHIN. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[BG97] L. BERTINI and G. GIACOMIN. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* **183**, no. 3, (1997), 571–607.

[BPRS93] L. BERTINI, E. PRESUTTI, B. RÜDIGER, and E. SAADA. Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic PDE. *Teor. Veroyatnost. i Primenen.* **38**, no. 4, (1993), 689–741. doi:10.1137/1138062.

[BR84] T. BROX and H. ROST. Equilibrium fluctuations of stochastic particle systems: the role of conserved quantities. *Ann. Probab.* **12**, no. 3, (1984), 742–759.

[Bre11] H. BREZIS. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[BZk97] A. BOVIER and M. S. ZAHRADNÍ K. The low-temperature phase of Kac-Ising models. *J. Statist. Phys.* **87**, no. 1-2, (1997), 311–332. doi:10.1007/BF02181490.

[CLO01] C.-C. CHANG, C. LANDIM, and S. OLLA. Equilibrium fluctuations of asymmetric simple exclusion processes in dimension $d \geq 3$. *Probab. Theory Related Fields* **119**, no. 3, (2001), 381–409. doi:10.1007/PL00008764.

[CMP95] M. CASSANDRO, R. MARRA, and E. PRESUTTI. Corrections to the critical temperature in 2d ising systems with kac potentials. *Journal of statistical physics* **78**, no. 3, (1995), 1131–1138.

[CMP97] M. CASSANDRO, R. MARRA, and E. PRESUTTI. Upper bounds on the critical temperature for Kac potentials. *J. Statist. Phys.* **88**, no. 3-4, (1997), 537–566. doi:10.1023/B:JOSS.0000015163.27899.8f.

[CW01] C. CARDON-WEBER. Cahn-Hilliard stochastic equation: existence of the solution and of its density. *Bernoulli* **7**, no. 5, (2001), 777–816.

[DG74] G. DEL GROSSO. On the local central limit theorem for Gibbs processes. *Comm. Math. Phys.* **37**, (1974), 141–160.

[DMPS89] A. DE MASI, E. PRESUTTI, and E. SCACCIATELLI. The weakly asymmetric simple exclusion process. *Ann. Inst. H. Poincaré Probab. Statist.* **25**, no. 1, (1989), 1–38.

[DMPSW86] A. DE MASI, E. PRESUTTI, H. SPOHN, and W. D. WICK. Asymptotic equivalence of fluctuation fields for reversible exclusion processes with speed change. *Ann. Probab.* **14**, no. 2, (1986), 409–423.

[DPD96] G. DA PRATO and A. DEBUSSCHE. Stochastic Cahn-Hilliard equation. *Nonlinear Anal.* **26**, no. 2, (1996), 241–263.

[DPD03] G. DA PRATO and A. DEBUSSCHE. Strong solutions to the stochastic quantization equations. *Ann. Probab.* **31**, no. 4, (2003), 1900–1916.

[DPZ14] G. DA PRATO and J. ZABCZYK. *Stochastic equations in infinite dimensions*, vol. 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second ed., 2014.

[EH17] D. ERHARD and M. HAIRER. Discretisation of regularity structures. *arXiv preprint arXiv:1705.02836* (2017).

[FR95] J. FRITZ and B. RÜDIGER. Time dependent critical fluctuations of a one-dimensional local mean field model. *Probab. Theory Related Fields* **103**, no. 3, (1995), 381–407. doi:10.1007/BF01195480.

[Gia91] G. GIACOMIN. van der Waals limit and phase separation in a particle model with Kawasaki dynamics. *J. Statist. Phys.* **65**, no. 1-2, (1991), 217–234. doi:10.1007/BF01329857.

[GJ13a] P. C. GONÇALVES and M. JARA. Scaling limits of additive functionals of interacting particle systems. *Comm. Pure Appl. Math.* **66**, no. 5, (2013), 649–677. doi:10.1002/cpa.21441.

[GJ13b] M. GUBINELLI and M. JARA. Regularization by noise and stochastic Burgers equations. *Stoch. Partial Differ. Equ. Anal. Comput.* **1**, no. 2, (2013), 325–350. doi:10.1007/s40072-013-0011-5.

[GJ14] P. C. GONÇALVES and M. JARA. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.* **212**, no. 2, (2014), 597–644. doi:10.1007/s00205-013-0693-x.

[GJS15] P. C. GONÇALVES, M. JARA, and S. SETHURAMAN. A stochastic Burgers equation from a class of microscopic interactions. *Ann. Probab.* **43**, no. 1, (2015), 286–338. doi:10.1214/13-AOP878.

[GK85] K. GAWĘDZKI and A. KUPIAINEN. Massless lattice φ^4_4 theory: rigorous control of a renormalizable asymptotically free model. *Comm. Math. Phys.* **99**, no. 2, (1985), 197–252.

[GL97] G. GIACOMIN and J. L. LEBOWITZ. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys.* **87**, no. 1-2, (1997), 37–61. doi:10.1007/BF02181479.

[GLP99] G. GIACOMIN, J. L. LEBOWITZ, and E. PRESUTTI. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. In *Stochastic partial differential equations: six perspectives*, vol. 64 of *Math. Surveys Monogr.*, 107–152. Amer. Math. Soc., Providence, RI, 1999. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1090/surv/064/03.

[Gri64] R. B. GRIFFITHS. Peierls proof of spontaneous magnetization in a two-dimensional Ising ferromagnet. *Phys. Rev. (2)* **136**, (1964), A437–A439.

[Hai14] M. HAIRER. A theory of regularity structures. *Invent. Math.* **198**, no. 2, (2014), 269–504. arXiv:1303.5113. doi:10.1007/s00222-014-0505-4.

[HI18] M. HAIRER and M. IBERTI. Tightness of the ising–kac model on the two-dimensional torus. *Journal of Statistical Physics* **171**, no. 4, (2018), 632–655. doi:10.1007/s10955-018-2033-x.

[HM15] M. HAIRER and K. MATETSKI. Discretisations of rough stochastic pdes. *arXiv preprint arXiv:1511.06937* (2015).

[HW13] M. HAIRER and H. WEBER. Rough Burgers-like equations with multiplicative noise. *Probab. Theory Related Fields* **155**, no. 1-2, (2013), 71–126.

[Ibe17] M. IBERTI. Convergence of glauber dynamic on ising-like models with kac interaction to ϕ_2^{2n}. *arXiv preprint arXiv:1708.00948* (2017).

[JL91] G. JONA-LASINIO. Stochastic reaction diffusion equations and interacting particle systems. *Ann. Inst. H. Poincaré Phys. Théor.* **55**, no. 2, (1991), 751–758. Multiscale phenomena (São Paulo, 1990).

[KL99] C. KIPNIS and C. LANDIM. *Scaling limits of interacting particle systems*, vol. 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[KOV89] C. KIPNIS, S. OLLA, and S. R. S. VARADHAN. Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.* **42**, no. 2, (1989), 115–137.

[KPZ86] M. KARDAR, G. PARISI, and Y.-C. ZHANG. Dynamic scaling of growing interfaces. *Physical Review Letters* **56**, no. 9, (1986), 889.

[KUH63] M. KAC, G. E. UHLENBECK, and P. C. HEMMER. On the van der Waals theory of the vapor-liquid equilibrium. I. Discussion of a one-dimensional model. *J. Mathematical Phys.* **4**, (1963), 216–228. doi:10.1063/1.1703946.

[Leb74] J. L. LEBOWITZ. GHS and other inequalities. *Comm. Math. Phys.* **35**, (1974), 87–92.

[Lig05] T. M. LIGGETT. *Interacting particle systems*. Classics in Mathematics. Springer-Verlag, Berlin, 2005. Reprint of the 1985 original.

[LOP91] J. L. LEBOWITZ, E. ORLANDI, and E. PRESUTTI. A particle model for spinodal decomposition. *J. Statist. Phys.* **63**, no. 5-6, (1991), 933–974. doi:10.1007/BF01029992.

[LP66] J. L. LEBOWITZ and O. PENROSE. Rigorous Treatment of the Van Der Waals-Maxwell Theory of the Liquid-Vapor Transition. *Journal of Mathematical Physics* **7**, (1966), 98–113. doi:10.1063/1.1704821.

[LY93] S. L. LU and H.-T. YAU. Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Comm. Math. Phys.* **156**, no. 2, (1993), 399–433.

[MW17a] J.-C. MOURRAT and H. WEBER. Convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2. *Comm. Pure Appl. Math.* **70**, no. 4, (2017), 717–812. doi:10.1002/cpa.21655.

[MW17b] J.-C. MOURRAT and H. WEBER. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.* **45**, no. 4, (2017), 2398–2476.

[Pen91] O. PENROSE. A mean-field equation of motion for the dynamic Ising model. *J. Statist. Phys.* **63**, no. 5-6, (1991), 975–986. doi:10.1007/BF01029993.

[Pre09] E. PRESUTTI. *Scaling limits in statistical mechanics and microstructures in continuum mechanics*. Theoretical and Mathematical Physics. Springer, Berlin, 2009.

[Pro90] P. PROTTER. *Stochastic integration and differential equations*, vol. 21 of *Applications of Mathematics (New York)*, x+302. Springer-Verlag, Berlin, 1990. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1007/978-3-662-02619-9. A new approach.

[Qua96] J. QUASTEL. Diffusion in disordered media. In *Nonlinear stochastic PDEs (Minneapolis, MN, 1994)*, vol. 77 of *IMA Vol. Math. Appl.*, 65–79. Springer, New York, 1996. URL http://0-dx.doi.org.pugwash.lib.warwick.ac.uk/10.1007/978-1-4613-8468-7_4.

[Rez94] F. REZAKHANLOU. Propagation of chaos for symmetric simple exclusions. *Comm. Pure Appl. Math.* **47**, no. 7, (1994), 943–957.

[SG73] B. SIMON and R. B. GRIFFITHS. The $(\phi^4)_2$ field theory as a classical Ising model. *Comm. Math. Phys.* **33**, (1973), 145–164.

[Spo91] H. SPOHN. *Large scale dynamics of interacting particles*. Texts and monographs in physics. Springer-Verlag, 1991.

[SW16] H. SHEN and H. WEBER. Glauber dynamics of 2d kac-blume-capel model and their stochastic pde limits. *arXiv preprint arXiv:1608.06556* (2016).

[SY95] H. SPOHN and H.-T. YAU. Bulk diffusivity of lattice gases close to criticality. *Journal of statistical physics* **79**, no. 1, (1995), 231–241.

[TW16] P. TSATSOULIS and H. WEBER. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *arXiv preprint arXiv:1609.08447* (2016).

[VY97] S. R. S. VARADHAN and H.-T. YAU. Diffusive limit of lattice gas with mixing conditions. *Asian J. Math.* **1**, no. 4, (1997), 623–678. doi:10.4310/AJM.1997.v1.n4.a1.